EQUATIONAL THEORIES OF SEMIGROUPS WITH ENRICHED SIGNATURE

K. AUINGER, I. DOLINKA, AND M. V. VOLKOV

Abstract. We present sufficient conditions for a unary semigroup variety to have no finite basis for its equational theory. In particular, we exhibit a 6-element involutory semigroup which is inherently non-finitely based as a unary semigroup. As applications we get several naturally arising unary semigroups without finite identity bases, for example: the semigroup of all complex 2×2-matrices endowed with Moore-Penrose inversion; the semigroup of all $n \times n$-matrices ($n \geq 2$) endowed with transposition over either a finite field or the Boolean semiring; various partition semigroups endowed with their natural involution, including the full partition semigroup $\mathfrak{C}_n$ for $n \geq 2$, the Brauer semigroup $\mathfrak{B}_n$ for $n \geq 4$ and the annular semigroup $\mathfrak{A}_n$ for $n \geq 4$, n even or a prime power. We also show that similar techniques apply to the finite basis problem for existence varieties of locally inverse semigroups.

Contents

Background and motivation

A fundamental and widely studied question connected with an algebraic structure is whether its equational theory is finitely axiomatizable. Being very natural by itself, this question, which is usually referred to as the *finite basis problem*, has also revealed a number of interesting and unexpected relations to many issues of theoretical and practical importance ranging from feasible algorithms for membership in certain classes of formal languages (see [1]) to classical number-theoretic conjectures (such as the Twin Prime, Goldbach, existence of odd perfect numbers and the infinitude of even perfect numbers: see [35] where it is shown that each of these conjectures is equivalent to the finite axiomatizability of the equational theory of a particular groupoid).

For the class of semigroups, an overview of results (up to the year 2000) concerning the finite basis problem can be found in the survey article [44] by the third author. In that article, the reader will find several **methods** to treat the problem. A very powerful one is to apply Sapir's classification of inherently non-finitely based finite semigroups [37,38], where a finite semigroup is called *inherently non-finitely based* if it is not contained in any finitely based locally finite variety. For example, a consequence of this classification is that the *Brandt monoid* B_2^1 formed by the 6 matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

under the usual matrix multiplication is inherently non-finitely based, and so, in particular, each finitely generated variety containing B_2^1 is not finitely based. Now observe that the monoid B_2^1 admits a fairly natural unary operation (transposition) and as such is an *inverse* semigroup. However Sapir's results do not apply to this extended signature. Even though B_2^1 is not finitely based as an inverse semigroup [19], it was shown by Sapir [40] that B_2^1 is **not** inherently non-finitely based as an inverse semigroup. More generally, Sapir showed that there do not at all exist inherently non-finitely based finite inverse semigroups. From the results of Margolis and Sapir [25] it follows that this is even true in the more general context of regular $*$-semigroups (see Remark 2.4 below).

Here is the starting point of the present paper: we shall treat the finite basis problem for **unary semigroups** where a *unary semigroup* is merely a semigroup equipped with some additional unary operation. The class of all groups forms a very natural and very important example of a unary semigroup variety, and many varieties of unary semigroups studied in literature have arisen as generalizations of this, for example, inverse semigroups or completely regular semigroups. On the other hand, there are several important individual semigroups that are endowed with a unary operation in a natural way: the semigroup $\mathrm{M}_n(\mathbb{C})$ of all complex $n \times n$-matrices admits several unary operations, for instance, complex-conjugate transposition *

and Moore-Penrose inversion †; the semigroup B_X of all binary relations on a non-empty set X admits the unary operation of taking the dual relation.

In this paper we adapt to the unary environment two classical approaches to the finite basis problem, namely, the method of inherently non-finitely based semigroups and 'the critical semigroup method' of the survey [44]. This leads to two fairly general and easy-to-verify conditions on a unary semigroup variety $\mathbf{V}$ guaranteeing that $\mathbf{V}$ is not finitely based. We present the corresponding results in Section 1. The rest of the paper is organized as follows. Section 2 collects several applications to the finite basis problem for unary semigroups of matrices; in particular, we consider the unary semigroups mentioned in the previous paragraph. Both methods of Section 1 are used here, and it turns out that they in some sense complement each other. Section 3 is devoted to a treatment of various partition semigroups (such as the Brauer semigroup, see [6]) that arose in representation theory and gained much attention recently. Here, as follows from an observation made in Section 2, the method of inherently non-finitely based semigroups does not apply but the 'critical semigroup method' works quite well. Section 4 gathers some further applications of the latter method, and, in Section 5, we demonstrate how the same approach applies to so-called existence varieties of locally inverse semigroups.

We assume the reader's acquaintance with basic concepts of the theory of varieties such as the HSP-theorem, see, e.g., [8, Chapter II]. As far as semigroup notions are concerned, we adopt the standard terminology and notation from [9]. It should be noted, however, that until the last section the presentation is to a reasonable extent self-contained so that most of the material in Sections 1–4 should be accessible to readers with very basic semigroup-theoretic background (such as some knowledge of Green's relations $\mathscr{D}$, $\mathscr{R}$ and $\mathscr{H}$, cf. [9, Section 2.1]).

1. Tools for proving the absence of a finite identity basis

1.1. **Preliminaries.** As mentioned, by a *unary semigroup* we mean an algebraic structure $\mathcal{S} = \langle S, \cdot, ^*\rangle$ of type $(2,1)$ such that the binary operation $\cdot$ is associative, i. e. $\langle S, \cdot\rangle$ is a semigroup. In general, we do not assume any additional identities involving the unary operation *. If the identities $(xy)^* = y^*x^*$ and $(x^*)^* = x$ happen to hold in $\mathcal{S}$, in other words, if the unary operation $x \mapsto x^*$ is an involutory anti-automorphism of the semigroup $\langle S, \cdot\rangle$, we call $\mathcal{S}$ an *involutory semigroup*. If, in addition, the identity $x = xx^*x$ holds, $\mathcal{S}$ is said to be a *regular $*$-semigroup*. Each group, subject to its inverse operation $x \mapsto x^{-1}$ is an involutory semigroup, even a regular $*$-semigroup; throughout the paper, any group is considered as a unary semigroup with respect to this inverse unary operation.

A wealth of examples of involutory semigroups and regular $*$-semigroups can be obtained via the following 'unary' version of the well known Rees matrix construction (see [9, Section 3.1] for a description of the construction

in the plain semigroup case). Let $\mathcal{G} = \langle G, \cdot, {}^{-1}\rangle$ be a group, 0 a symbol beyond G, and I a non-empty set. We formally set $0^{-1} = 0$. Given an $I \times I$-matrix $P = (p_{ij})$ over $G \cup \{0\}$ such that $p_{ij} = p_{ji}^{-1}$ for all $i, j \in I$, we define a multiplication $\cdot$ and a unary operation * on the set $(I \times G \times I) \cup \{0\}$ by the following rules:

$$a \cdot 0 = 0 \cdot a = 0 \quad \text{for all } a \in (I \times G \times I) \cup \{0\},$$
$$(i, g, j) \cdot (k, h, \ell) = \begin{cases} (i, gp_{jk}h, \ell) & \text{if } p_{jk} \neq 0, \\ 0 & \text{if } p_{jk} = 0; \end{cases}$$
$$(i, g, j)^* = (j, g^{-1}, i), \ 0^* = 0.$$

It can be easily checked that $\langle (I \times G \times I) \cup \{0\}, \cdot, {}^*\rangle$ becomes an involutory semigroup; it will be a regular $*$-semigroup precisely when $p_{ii} = e$ (the identity element of the group $\mathcal{G}$) for all $i \in I$. We denote this unary semigroup by $\mathcal{M}^0(I, \mathcal{G}, I; P)$ and call it the *unary Rees matrix semigroup over* $\mathcal{G}$ *with the sandwich matrix* P. If the involved group $\mathcal{G}$ happens to be the trivial group $\mathcal{E} = \{e\}$ then we usually shall ignore the group entry and represent the non-zero elements of such a Rees matrix semigroup by the pairs (i, j) with $i, j \in I$.

In this paper, the 10-element unary Rees matrix semigroup over the trivial group $\mathcal{E} = \{e\}$ with the sandwich matrix

$$\begin{pmatrix} e & e & e \\ e & e & 0 \\ e & 0 & e \end{pmatrix}$$

plays a key role; we denote this semigroup by $\mathcal{K}_3$. Thus, subject to the convention mentioned above, $\mathcal{K}_3$ consists of the nine pairs (i, j), $i, j \in \{1, 2, 3\}$, and the element 0, and the operations restricted to its non-zero elements can be described as follows:

$$(i, j) \cdot (k, \ell) = \begin{cases} (i, \ell) & \text{if } (j, k) \neq (2, 3), (3, 2), \\ 0 & \text{otherwise}; \end{cases} \tag{1.1}$$
$$(i, j)^* = (j, i).$$

Another unary semigroup that will be quite useful in the sequel is the *free involutory semigroup* $\mathrm{FI}(X)$ on a given alphabet X. It can be constructed as follows. Let $\overline{X} = \{x^* \mid x \in X\}$ be a disjoint copy of X and define $(x^*)^* = x$ for all $x^* \in \overline{X}$. Then $\mathrm{FI}(X)$ is the free semigroup $(X \cup \overline{X})^+$ endowed with an involution * defined by

$$(x_1 \cdots x_m)^* = x_m^* \cdots x_1^*$$

for all $x_1, \ldots, x_m \in X \cup \overline{X}$. We will refer to elements of $\mathrm{FI}(X)$ as to *involutory words over* X while elements of the free semigroup X^+ will be referred to as (plain semigroup) words over X.

1.2. **A unary version of the critical semigroup method.** The formulation of our first main result involves two simple operators on unary semigroup varieties. For any unary semigroup $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ we denote by $\mathrm{H}(\mathcal{S})$ the unary subsemigroup of $\mathcal{S}$ which is generated by all elements of the form xx^*, where $x \in S$. We call $\mathrm{H}(\mathcal{S})$ the *Hermitian subsemigroup* of $\mathcal{S}$. For any variety $\mathbf{V}$ of unary semigroups, let $\mathrm{H}(\mathbf{V})$ be the subvariety of $\mathbf{V}$ generated by all Hermitian subsemigroups of members of $\mathbf{V}$. Likewise, given a positive integer n, let $\mathrm{P}_n(\mathcal{S})$ be the unary subsemigroup of $\mathcal{S}$ which is generated by all elements of the form x^n, where $x \in S$, and let $\mathrm{P}_n(\mathbf{V})$ be the subvariety of $\mathbf{V}$ generated by all subsemigroups $\mathrm{P}_n(\mathcal{S})$, where $\mathcal{S} \in \mathbf{V}$.

Denote by $\mathsf{var}\,\mathcal{S}$ the variety generated by a given unary semigroup $\mathcal{S}$. The following easy observation will be useful in the sequel as it helps calculating the effect of the operators H and P_n.

Lemma 1.1. $\mathrm{H}(\mathsf{var}\,\mathcal{S}) = \mathsf{var}\,\mathrm{H}(\mathcal{S})$ *and* $\mathrm{P}_n(\mathsf{var}\,\mathcal{S}) = \mathsf{var}\,\mathrm{P}_n(\mathcal{S})$ *for every unary semigroup* $\mathcal{S}$ *and for each* $n \in \mathbb{N}$.

Proof. The non-trivial part of the first claim is the inclusion $\mathrm{H}(\mathsf{var}\,\mathcal{S}) \subseteq \mathsf{var}\,\mathrm{H}(\mathcal{S})$. Let $\mathcal{T} \in \mathsf{var}\,\mathcal{S}$, then $\mathcal{T}$ is a homomorphic image of a unary subsemigroup $\mathcal{U}$ of a direct product of several copies of $\mathcal{S}$. But then $\mathrm{H}(\mathcal{T})$ is a homomorphic image of $\mathrm{H}(\mathcal{U})$. As is easy to see, $\mathrm{H}(\mathcal{U})$ is a unary subsemigroup of a direct product of several copies of $\mathrm{H}(\mathcal{S})$. Thus $\mathrm{H}(\mathcal{T}) \in \mathsf{var}\,\mathrm{H}(\mathcal{S})$. Since this holds for an arbitrary $\mathcal{T} \in \mathsf{var}\,\mathcal{S}$, we conclude that $\mathrm{H}(\mathsf{var}\,\mathcal{S}) \subseteq \mathsf{var}\,\mathrm{H}(\mathcal{S})$. The second assertion can be treated in a completely similar way. □

We are now ready to state our first main result.

Theorem 1.2. *Let* $\mathbf{V}$ *be any unary semigroup variety such that* $\mathcal{K}_3 \in \mathbf{V}$. *If either*

- *there exists a group* $\mathcal{G}$ *such that* $\mathcal{G} \in \mathbf{V}$ *but* $\mathcal{G} \notin \mathrm{H}(\mathbf{V})$

or

- *there exist a positive integer* d *and a group* $\mathcal{G}$ *of exponent dividing* d *such that* $\mathcal{G} \in \mathbf{V}$ *but* $\mathcal{G} \notin \mathrm{P}_d(\mathbf{V})$,

then $\mathbf{V}$ *has no finite basis of identities.*

Proof. Assume first that there exists a group $\mathcal{G} \in \mathbf{V}$ for which $\mathcal{G} \notin \mathrm{H}(\mathbf{V})$.

1. First we recall the basic idea of 'the critical semigroup method' in the unary setting. Suppose that $\mathbf{V}$ is finitely based. If Σ is a finite identity basis of the variety $\mathbf{V}$ then there exists a positive integer ℓ such that all identities from Σ depend on at most ℓ letters. Therefore identities from Σ hold in a unary semigroup $\mathcal{S}$ whenever all ℓ-generated unary subsemigroups of $\mathcal{S}$ satisfy Σ. In other words, $\mathcal{S}$ belongs to $\mathbf{V}$ whenever all of its ℓ-generated unary subsemigroups are in $\mathbf{V}$. We see that in order to prove our theorem it is sufficient to construct, for any given positive integer k, a unary semigroup $\mathcal{T}_k \notin \mathbf{V}$ for which all k-generated unary subsemigroups of $\mathcal{T}_k$ belong to $\mathbf{V}$.

2. Fix an identity $u(x_1, \dots, x_m) = v(x_1, \dots, x_m)$ that holds in $\mathrm{H}(\mathbf{V})$ but fails in the group $\mathcal{G}$. The latter means that, for some $g_1, \dots, g_m \in G$,

substitution of g_i for x_i yields

$$u(g_1,\dots,g_m) \neq v(g_1,\dots,g_m). \tag{1.2}$$

Now, for each positive integer k, let $n = \max\{4, 2k+1\}$, $I = \{1,\dots,nm\}$ and consider the unary Rees matrix semigroup $\mathcal{T}_k = \mathcal{M}^0(I,\mathcal{G},I;P_k)$ over the group $\mathcal{G}$ with the sandwich matrix

$$P_k = \begin{pmatrix} M_n(g_1) & E_n & O_n & O_n & \cdots & O_n & E_n^T \\ E_n^T & M_n(g_2) & E_n & O_n & \cdots & O_n & O_n \\ O_n & E_n^T & M_n(g_3) & E_n & \cdots & O_n & O_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O_n & O_n & O_n & O_n & \cdots & E_n & O_n \\ O_n & O_n & O_n & O_n & \cdots & M_n(g_{m-1}) & E_n \\ E_n & O_n & O_n & O_n & \cdots & E_n^T & M_n(g_m) \end{pmatrix},$$

where O_n is the zero $n\times n$-matrix, E_n is the $n\times n$-matrix having e (the identity of $\mathcal{G}$) in the position $(n,1)$ and 0 in all other positions, E_n^T is the transpose of E_n, and $M_n(g)$ denotes the $n\times n$-matrix of the form

$$M_n(g) = \begin{pmatrix} e & g & 0 & \cdots & 0 & 0 & e \\ g^{-1} & e & e & \cdots & 0 & 0 & 0 \\ 0 & e & e & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e & e & 0 \\ 0 & 0 & 0 & \cdots & e & e & e \\ e & 0 & 0 & \cdots & 0 & e & e \end{pmatrix}.$$

(This construction is in a sense a combination of those of the first and the third authors' papers [3] and [43].) We are going to prove that $\mathcal{T}_k$ enjoys the two properties needed, namely, it does not belong to $\mathbf{V}$, but each k-generated unary subsemigroup of $\mathcal{T}_k$ lies in $\mathbf{V}$.

3. In order to prove that $\mathcal{T}_k \notin \mathbf{V}$, we construct an identity that holds in $\mathbf{V}$, but fails in $\mathcal{T}_k$. Consider the following m terms in mn letters $x_1,\dots,x_{mn}$:

$$\begin{aligned}
w_1 &= [x_1x_1^*\cdots x_nx_n^*][(x_{n+1}\cdots x_{2n})(x_{n+1}\cdots x_{2n})^*x_{2n}x_{2n}^*]\cdots \\
&\qquad [(x_{(m-1)n+1}\cdots x_{mn})(x_{(m-1)n+1}\cdots x_{mn})^*x_{mn}x_{mn}^*],\\
w_2 &= [(x_1\cdots x_n)(x_1\cdots x_n)^*x_nx_n^*][x_{n+1}x_{n+1}^*\cdots x_{2n}x_{2n}^*]\times \\
&\qquad [(x_{2n+1}\cdots x_{3n})(x_{2n+1}\cdots x_{3n})^*x_{3n}x_{3n}^*]\cdots \\
&\qquad [(x_{(m-1)n+1}\cdots x_{mn})(x_{(m-1)n+1}\cdots x_{mn})^*x_{mn}x_{mn}^*],\\
w_3 &= [(x_1\cdots x_n)(x_1\cdots x_n)^*x_nx_n^*][(x_{n+1}\cdots x_{2n})(x_{n+1}\cdots x_{2n})^*x_{2n}x_{2n}^*]\times \\
&\qquad [x_{2n+1}x_{2n+1}^*\cdots x_{3n}x_{3n}^*]\cdots[(x_{(m-1)n+1}\dots x_{mn})(x_{(m-1)n+1}\dots x_{mn})^*x_{mn}x_{mn}^*],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
w_m &= [(x_1\cdots x_n)(x_1\cdots x_n)^*x_nx_n^*][(x_{n+1}\cdots x_{2n})(x_{n+1}\cdots x_{2n})^*x_{2n}x_{2n}^*]\dots \\
&\qquad [(x_{(m-2)n+1}\cdots x_{(m-1)n})(x_{(m-2)n+1}\cdots x_{(m-1)n})^*x_{(m-1)n}x_{(m-1)n}^*]\times \\
&\qquad [x_{(m-1)n+1}x_{(m-1)n+1}^*\cdots x_{mn}x_{mn}^*].
\end{aligned}$$

Substituting w_i for x_i in u respectively v, we get the identity

$$u(w_1, \dots, w_m) = v(w_1, \dots, w_m) \tag{1.3}$$

which holds in the variety $\mathbf{V}$. Indeed, if we take any $\mathcal{S} \in \mathbf{V}$, then, since $ss^* \in \mathrm{H}(\mathcal{S})$ for any $s \in \mathcal{S}$, all the values of w_i belong to the Hermitian subsemigroup $\mathrm{H}(\mathcal{S})$ of $\mathcal{S}$. This subsemigroup, however, lies in $\mathrm{H}(\mathbf{V})$, and therefore, satisfies the identity $u = v$.

Now we shall show that (1.3) fails in $\mathcal{T}_k$. Indeed, substituting $(i, e, i) \in \mathcal{T}_k$ for x_i, we calculate that the value of every term of the form

$$(x_{(j-1)n+1} \cdots x_{jn})(x_{(j-1)n+1} \cdots x_{jn})^* x_{jn} x_{jn}^*$$

is equal to $((j-1)n+1, e, jn)$ while the value of each term of the form

$$x_{(j-1)n+1} x_{(j-1)n+1}^* \cdots x_{jn} x_{jn}^*$$

is equal to $((j-1)n+1, g_j, jn)$. Hence the value of w_j is just $(1, g_j, mn)$. Therefore, under this substitution, the left hand part of (1.3) takes the value $(s, u(g_1, \dots, g_m), t)$ for suitable $s, t \in \{1, mn\}$ while the value of the right hand part of (1.3) is $(s', v(g_1, \dots, g_m), t')$ (again for suitable $s', t' \in \{1, mn\}$). In view of the inequality (1.2), these elements do not coincide in $\mathcal{T}_k$.

4. It remains to prove that each k-generated unary subsemigroup of $\mathcal{T}_k$ lies in $\mathbf{V}$. For every m-tuple $(\lambda_1, \dots, \lambda_m)$ of positive integers satisfying

$$1 \le \lambda_1 \le n < \lambda_2 \le 2n < \lambda_3 \le \dots (m-1)n < \lambda_m \le mn, \tag{1.4}$$

consider the unary subsemigroup $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ of $\mathcal{T}_k$ consisting of 0 and all triples (i, g, j) such that $g \in \mathcal{G}$ and $i, j \notin \{\lambda_1, \dots, \lambda_m\}$. Using that $2k < n$ according to our choice of n, one concludes that any given k elements of $\mathcal{T}_k$ must be contained in $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ for suitable $\lambda_1, \dots, \lambda_m$. Thus it is sufficient to prove that each semigroup of the form $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ belongs to the variety $\mathbf{V}$.

Let us fix positive integers $\lambda_1, \dots, \lambda_m$ satisfying (1.4). When multiplying triples from $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$, the $\lambda_1^{\text{th}}, \dots, \lambda_m^{\text{th}}$ rows and columns of the sandwich matrix P_k are never involved. Therefore we can identify $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ with the unary Rees matrix semigroup $\mathcal{M}^0(I', \mathcal{G}, I'; P_k')$ over the group $\mathcal{G}$ where $I' = I \setminus \{\lambda_1, \dots, \lambda_m\}$ and the sandwich matrix $P_k' = P_k(\lambda_1, \dots, \lambda_m)$ is obtained from P_k by deleting its $\lambda_1^{\text{th}}, \dots, \lambda_m^{\text{th}}$ rows and columns. Note that by (1.4) exactly one row and one column of each block $M_n(g_i)$ is deleted.

Now we transform the matrix $P_k(\lambda_1, \dots, \lambda_m)$ as follows. For each i such that $(i-1)n+2 < \lambda_i$, we multiply successively

$$\begin{array}{l}\text{the row } ((i-1)n+2) \text{ by } g_i \text{ from the left and}\\ \text{the column } ((i-1)n+2) \text{ by } g_i^{-1} \text{ from the right;}\\ \text{the row } ((i-1)n+3) \text{ by } g_i \text{ from the left and}\\ \text{the column } ((i-1)n+3) \text{ by } g_i^{-1} \text{ from the right;}\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \text{the row } (\lambda_i-1) \text{ by } g_i \text{ from the left and}\\ \text{the column } (\lambda_i-1) \text{ by } g_i^{-1} \text{ from the right.}\end{array} \tag{1.5}$$

In order to help the reader to understand the effect of the transformations (1.5), we illustrate their action on the block obtained from $M_n(g_i)$ by removing the λ_i^{th} row and column in the following scheme in which λ_i has been chosen to be equal to $(i-1)n+5$. (The transformations have no effect beyond $M_n(g_i)$ because all the rows and columns of $P_k(\lambda_1, \dots, \lambda_m)$ involved in (1.5) have non-zero entries only within $M_n(g_i)$.)

The block obtained from $M_n(g_i)$ by erasing the $((i-1)n+5)^{\text{th}}$ row and column

$$\begin{pmatrix} e & g_i & 0 & 0 & 0 & \cdots & 0 & e\\ g_i^{-1} & e & e & 0 & 0 & \cdots & 0 & 0\\ 0 & e & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & e & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & e & e\\ e & 0 & 0 & 0 & 0 & \cdots & e & e \end{pmatrix}$$

After the first transformation

$$\begin{pmatrix} e & e & 0 & 0 & 0 & \cdots & 0 & e\\ e & e & g_i & 0 & 0 & \cdots & 0 & 0\\ 0 & g_i^{-1} & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & e & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & e & e\\ e & 0 & 0 & 0 & 0 & \cdots & e & e \end{pmatrix}$$

After the second transformation

$$\begin{pmatrix} e & e & 0 & 0 & 0 & \cdots & 0 & e\\ e & e & e & 0 & 0 & \cdots & 0 & 0\\ 0 & e & e & g_i & 0 & \cdots & 0 & 0\\ 0 & 0 & g_i^{-1} & e & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & e & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & e & e\\ e & 0 & 0 & 0 & 0 & \cdots & e & e \end{pmatrix}$$

After the third transformation

$$\begin{pmatrix} e & e & 0 & 0 & 0 & \cdots & 0 & e\\ e & e & e & 0 & 0 & \cdots & 0 & 0\\ 0 & e & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & e & e & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & e & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & e & e\\ e & 0 & 0 & 0 & 0 & \cdots & e & e \end{pmatrix}$$

Now it should be clear that also in general the transformations (1.5) result in a matrix Q_k all of whose non-zero entries are equal to e. On the other hand, it is known (see, e. g., [3, Proposition 6.2]) that the transformations (1.5) of the sandwich matrix do not change the unary semigroup $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$; in other words $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ is isomorphic to the $I' \times I'$ unary Rees matrix semigroup $\mathcal{R}_k$ over $\mathcal{G}$ with the sandwich matrix Q_k. Let $\mathcal{U}_k$ be the $I' \times I'$

unary Rees matrix semigroup over the trivial group $\mathcal{E}$ with the sandwich matrix Q_k. It is easy to check that the mapping $\mathcal{G} \times \mathcal{U}_k \to \mathcal{R}_k$ defined by

$$(g, (i, e, j)) \mapsto (i, g, j), \quad (g, 0) \mapsto 0$$

for all $g \in \mathcal{G}$, $i, j \in I'$, is a unary semigroup homomorphism onto $\mathcal{R}_k$. Now we note that $\mathcal{G} \in \mathbf{V}$ and $\mathcal{U}_k$ belongs to the variety generated by $\mathcal{K}_3$ (see [3, Theorem 5.2]). This yields $\mathcal{T}_k(\lambda_1, \dots, \lambda_m) \cong \mathcal{R}_k \in \mathbf{V}$.

The case when there exist a positive integer d and a group $\mathcal{G}$ of exponent dividing d such that $\mathcal{G} \in \mathbf{V}$ but $\mathcal{G} \notin \mathrm{P}_d(\mathbf{V})$ can be treated in a very similar way. The construction of the critical semigroups remains the same, and the only modification to be made in the rest of the proof is to replace the terms w_i above by the following plain semigroup words:

$w_1 = [x_1^d \cdots x_n^d][x_{n+1} \cdots x_{2n}]^d \cdots [x_{(m-1)n+1} \cdots x_{mn}]^d,$
$w_2 = [x_1 \cdots x_n]^d[x_{n+1}^d \cdots x_{2n}^d] \cdots [x_{(m-1)n+1} \cdots x_{mn}]^d,$
....
$w_m = [x_1 \cdots x_n]^d[x_{n+1} \cdots x_{2n}]^d \cdots [x_{(m-1)n+1}^d \cdots x_{mn}^d].$

(These words already have been used in the plain semigroup case by the third author [43].) □

1.3. **A unary version of the method of inherently non-finitely based semigroups.** Recall that a finite algebraic structure of type τ is called *inherently non-finitely based* if it is not contained in any finitely based locally finite variety of type τ. Here we prove a sufficient condition for an involutory semigroup to be inherently non-finitely based and exhibit two concrete examples of involutory semigroups satisfying this condition. These examples will be essentially used at later stages of the paper.

Let $x_1, x_2, \dots, x_n, \dots$ be a sequence of letters. The sequence $\{Z_n\}_{n=1,2,\dots}$ of *Zimin words* is defined inductively by $Z_1 = x_1$, $Z_{n+1} = Z_n x_{n+1} Z_n$. We say that an involutory word v is an *involutory isoterm for a unary semigroup* $\mathcal{S}$ if the only involutory word v' such that $\mathcal{S}$ satisfies the involutory semigroup identity $v = v'$ is the word v itself.

Theorem 1.3. *Let $\mathcal{S}$ be a finite involutory semigroup. If all Zimin words are involutory isoterms for $\mathcal{S}$, then $\mathcal{S}$ is inherently non-finitely based.*

Proof. Arguing by contradiction, suppose that $\mathcal{S}$ belongs to a finitely based locally finite variety $\mathbf{V}$. If Σ is a finite identity basis of $\mathbf{V}$, then there exists a positive integer ℓ such that all identities from Σ depend on at most ℓ letters. Clearly, all identities in Σ hold in $\mathcal{S}$. In the following, our aim will be to construct, for any given positive integer k, an infinite, finitely generated involutory semigroup $\mathcal{T}_k$ which satisfies all identities in at most k variables that hold in $\mathcal{S}$. In particular, $\mathcal{T}_k$ will satisfy all identities from Σ. This yields a contradiction, as then we must conclude that $\mathcal{T}_\ell \in \mathbf{V}$, which is impossible by the local finiteness of $\mathbf{V}$.

We shall employ a construction invented by Sapir [38], see also his lecture notes [41]. We fix k and let $r = 6k + 2$. Consider the $r^2 \times r$-matrix M

shown in Fig. 1 on the left. All odd columns of M are identical and equal

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r & \cdots & 1 & r \\ 2 & 1 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & r & \cdots & 2 & r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & 1 & \cdots & r & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & r & r \end{pmatrix} \qquad M_A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r-1} & a_{1r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{11} & a_{r2} & \cdots & a_{1r-1} & a_{rr} \\ a_{21} & a_{12} & \cdots & a_{2r-1} & a_{1r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{21} & a_{r2} & \cdots & a_{2r-1} & a_{rr} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{r1} & a_{12} & \cdots & a_{rr-1} & a_{1r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr-1} & a_{rr} \end{pmatrix}$$

FIGURE 1. The matrices M and M_A

to the transpose of the row $(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, r, r, \ldots, r)$ where each number occurs r times. All even columns of M are identical and equal to the transpose of the row $(1, 2, \ldots, r, 1, 2, \ldots, r, \ldots, 1, 2, \ldots, r)$ in which the block $1, 2, \ldots, r$ occurs r times.

Now consider the alphabet $A = \{a_{ij} \mid 1 \leq i, j \leq r\}$ of cardinality r^2. We convert the matrix M to the matrix M_A (shown in Fig. 1 on the right) by replacing numbers by letters according to the following rule: whenever the number i occurs in the column j of M, we substitute it with the letter a_{ij} to get the corresponding entry in M_A.

Let v_t be the word in the t^{th} row of the matrix M_A. Consider the endomorphism $\gamma : A^+ \to A^+$ defined by

$$\gamma(a_{ij}) = v_{(i-1)r+j}.$$

Let V_k be the set of all factors of the words in the sequence $\{\gamma^m(a_{11})\}_{m=1,2,\ldots}$ and let 0 be a symbol beyond V_k. We define a multiplication $\cdot$ on the set $V_k \cup \{0\}$ as follows:

$$u \cdot v = \begin{cases} uv & \text{if } u, v, uv \in V_k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\langle V_k \cup \{0\}, \cdot\rangle$ becomes a semigroup which we denote by $\mathcal{V}_k^0$. Using this semigroup, we can conveniently reformulate two major combinatorial results by Sapir:

Proposition 1.4. [41, Proposition 2.1] *Let $X_k = \{x_1, \ldots, x_k\}$ and $w \in X_k^+$. Assume that there exists a homomorphism $\varphi : X_k^+ \to \mathcal{V}_k^0$ for which $\varphi(w) \neq 0$. Then there is an endomorphism $\psi : X_k^+ \to X_k^+$ such that the word $\psi(w)$ appears as a factor in the Zimin word Z_k.*

Proposition 1.5. [41, Lemma 4.14] *Let $X_k = \{x_1, \ldots, x_k\}$ and $w, w' \in X_k^+$. Assume that there exists a homomorphism $\varphi : X_k^+ \to \mathcal{V}_k^0$ for which $\varphi(w) \neq \varphi(w')$. Then the identity $w = w'$ implies a non-trivial semigroup identity of the form $Z_{k+1} = z$.*

Now let $\overleftarrow{\mathcal{V}}_k^0$ denote the semigroup anti-isomorphic to $\mathcal{V}_k^0$; we shall use the notation $x \mapsto x^*$ for the mutual anti-isomorphisms between $\mathcal{V}_k^0$ and $\overleftarrow{\mathcal{V}}_k^0$ in both directions and denote $\{v^* \mid v \in V_k\}$ by $\overline{V}_k$. Let

$$\mathcal{T}_k = \langle V_k \cup \overline{V}_k \cup \{0\}, \cdot, {}^* \rangle$$

be the 0-direct union of $\mathcal{V}_k^0$ and $\overleftarrow{\mathcal{V}}_k^0$; this means that we identify 0 with 0^*, preserve the multiplication in both $\mathcal{V}_k^0$ and $\overleftarrow{\mathcal{V}}_k^0$, and set $u \cdot v^* = u^* \cdot v = 0$ for all $u, v \in V_k$. This is the unary semigroup we need.

It is clear that $\mathcal{T}_k$ is infinite and is generated (as a unary semigroup) by the set A which is finite. It remains to verify that $\mathcal{T}_k$ satisfies every identity in at most k variables that holds in our initial unary semigroup $\mathcal{S}$. So, let $p, q \in \mathrm{FI}(X_k)$ and suppose that the identity $p = q$ holds in $\mathcal{S}$ but fails in $\mathcal{T}_k$. Then there exists a unary semigroup homomorphism $\varphi : \mathrm{FI}(X_k) \to \mathcal{T}_k$ for which $\varphi(p) \neq \varphi(q)$. Hence, at least one of the elements $\varphi(p)$ and $\varphi(q)$ is not equal to 0; (without loss of generality) assume that $\varphi(p) \neq 0$. Then we may also assume $\varphi(p) \in V_k$; otherwise we may consider the identity $p^* = q^*$ instead of $p = q$. Since $\varphi(p) \neq 0$, there is no letter $x \in X_k$ such that p contains both x and x^*. Now we define a substitution $\sigma : \mathrm{FI}(X_k) \to \mathrm{FI}(X_k)$ as follows:

$$\sigma(x) = \begin{cases} x^* & \text{if } p \text{ contains } x^*, \\ x & \text{otherwise.} \end{cases}$$

Then $\sigma(p)$ does not contain any starred letter, thus being a plain word in X_k^+. Since σ^2 is the identity mapping, we have $\varphi(p) = (\varphi\sigma)(\sigma(p))$, and $\varphi\sigma$ maps X_k^+ into $\mathcal{V}_k^0$. Now we consider two cases.

Case 1: *$\sigma(q)$ contains a starred letter.* We apply Proposition 1.4 to the plain word $\sigma(p)$ and the semigroup homomorphism $X_k^+ \to \mathcal{V}_k^0$ obtained by restricting $\varphi\sigma$ to X_k^+. We conclude that there is an endomorphism ψ of X_k^+ such that the word $\psi(\sigma(p))$ appears as a factor in the Zimin word Z_k. Thus, $Z_k = z'\psi(\sigma(p))z''$ for some z', z'' (that may be empty). The endomorphism ψ extends in a natural way to an endomorphism of the free involutory semigroup $\mathrm{FI}(X_k)$ and there is no harm in denoting the extension by ψ as well. The identity $p = q$ implies the identity

$$z'\psi(\sigma(p))z'' = z'\psi(\sigma(q))z''. \tag{1.6}$$

The left hand side of (1.6) is Z_k and the identity is not trivial because its right hand side involves a starred letter. Since $p = q$ holds in our initial semigroup $\mathcal{S}$, so does (1.6). But this contradicts the assumption that all Zimin words are involutory isoterms for $\mathcal{S}$.

Case 2: *$\sigma(q)$ contains no starred letter.* In this case $\sigma(q)$ is a plain word in X_k^+, and we are in a position to apply Proposition 1.5 to the semigroup

identity $\sigma(p) = \sigma(q)$ and the semigroup homomorphism $X_k^+ \to \mathcal{V}_k^0$ obtained by restricting $\varphi\sigma$ to X_k^+. We conclude that $\sigma(p) = \sigma(q)$ implies a non-trivial semigroup identity $Z_{k+1} = z$. Therefore the identity $p = q$ implies $Z_{k+1} = z$, and we again get a contradiction. $\square$

As a concrete example of an inherently non-finitely based involutory semigroup, consider the *twisted Brandt monoid* $\mathcal{TB}_2^1 = \langle B_2^1, \cdot, {}^* \rangle$, where $\langle B_2^1, \cdot \rangle$ is the usual 6-element Brandt monoid and the unary operation * fixes the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and swaps each of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the other one.

Corollary 1.6. *The twisted Brandt monoid $\mathcal{TB}_2^1$ is inherently non-finitely based.*

Proof. By Theorem 1.3 we only have to show that $\mathcal{TB}_2^1$ satisfies no non-trivial involutory semigroup identity of the form $Z_n = z$. If z is a plain semigroup word, we can refer to [38, Lemma 3.7] which shows that the Brandt monoid $\langle B_2^1, \cdot \rangle$ does not satisfy any non-trivial **semigroup** identity of the form $Z_n = z$. If we suppose that the involutory word z contains a starred letter, we can substitute the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all letters occurring in Z_n and z. Since this matrix is idempotent, the value of the word Z_n under this substitution equals $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. On the other hand, z evaluates to a product involving the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and it is easy to see that such a product is equal to either $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Thus, the identity $Z_n = z$ cannot hold in $\mathcal{TB}_2^1$ in this case as well. $\square$

An equivalent way to define $\mathcal{TB}_2^1$ is to consider the 5-element unary Rees matrix semigroup over the trivial group $\mathcal{E} = \{e\}$ with the sandwich matrix

$$\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$$

and then to adjoin to this unary Rees matrix semigroup an identity element. For convenience and later use we note that $\mathcal{TB}_2^1$ can thus be realized as the set

$$\{(1,1), (1,2), (2,1), (2,2), 0, 1\}$$

endowed with the operations

$$(i,j) \cdot (k,\ell) = \begin{cases} (i,\ell) & \text{if } (j,k) \in \{(1,2),(2,1)\}, \\ 0 & \text{otherwise;} \end{cases} \tag{1.7}$$

$$1 \cdot x = x = x \cdot 1,\ 0 \cdot x = 0 = x \cdot 0 \text{ for all } x;$$

$$(i,j)^* = (j,i),\ 1^* = 1,\ 0^* = 0.$$

Suppose that $\mathcal{A}$ is a finite algebraic structure for which the variety $\mathsf{var}\,\mathcal{A}$ contains an inherently non-finitely based algebraic structure. Immediately from the definition it follows that $\mathcal{A}$ is also inherently non-finitely based. This observation is useful, in particular, for the justification of our second example of an involutory inherently non-finitely based semigroup. This is a 'twisted version' $\mathcal{TA}_2^1$ of another 6-element monoid that often shows up under the name A_2^1 in the theory of semigroup varieties. The unary semigroup $\mathcal{TA}_2^1$ is formed by the 6 matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

under the usual matrix multiplication and the unary operation that swaps each of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

with the other one and fixes all other matrices. Alternatively, $\mathcal{TA}_2^1$ is obtained from the 5-element unary Rees matrix semigroup $\mathcal{A}_2$ over $\mathcal{E} = \{e\}$ with the sandwich matrix

$$\begin{pmatrix} 0 & e \\ e & e \end{pmatrix} \tag{1.8}$$

by adjoining an identity element. Again, for later use, we note that $\mathcal{TA}_2^1$ can be realized as the set

$$\{(1,1), (1,2), (2,1), (2,2), 0, 1\}$$

endowed with the operations

$$\begin{gathered} (i,j)\cdot(k,\ell) = \begin{cases} (i,\ell) & \text{if } (j,k) \neq (1,1) \\ 0 & \text{if } (j,k) = (1,1); \end{cases} \\ 1\cdot x = x = x\cdot 1,\ 0\cdot x = 0 = x\cdot 0 \text{ for all } x; \\ (i,j)^* = (j,i),\ 1^* = 1,\ 0^* = 0. \end{gathered} \tag{1.9}$$

Corollary 1.7. *The involutory monoid $\mathcal{TA}_2^1$ is inherently non-finitely based.*

Proof. We represent $\mathcal{TA}_2^1$ as in (1.9) and $\mathcal{TB}_2^1$ as in (1.7) and consider the direct square $\mathcal{TA}_2^1 \times \mathcal{TA}_2^1$. It is then easy to check that the twisted Brandt monoid $\mathcal{TB}_2^1$ is a homomorphic image of the unary subsemigroup of $\mathcal{TA}_2^1 \times \mathcal{TA}_2^1$ generated by the pairs $(1,1)$, $\big((1,1),(2,2)\big)$ and $\big((2,2),(1,1)\big)$. Thus, $\mathcal{TB}_2^1$ belongs to $\mathsf{var}\,\mathcal{TA}_2^1$. Since by Corollary 1.6 $\mathcal{TB}_2^1$ is inherently non-finitely based, so is $\mathcal{TA}_2^1$. □

Remark 1.1. Sapir [38, Proposition 7] has shown that a (plain) finite semigroup $\mathcal{S}$ is inherently non-finitely based **if and only if** all Zimin words are isoterms for $\mathcal{S}$, that is, $\mathcal{S}$ satisfies no non-trivial semigroup identity of the form $Z_n = z$. Our Theorem 1.3 models the 'if' part of this statement but we do not know whether or not the 'only if' part transfers to the involutory environment. Some partial results in this direction have been recently obtained by the second author [10].

2. Applications to matrix semigroups

2.1. **A property of matrices of rank 1.** For a field $\mathcal{K}$, let $\mathrm{M}_n(\mathcal{K})$ be the ring of all $n\times n$-matrices over $\mathcal{K}$. We start with registering a simple property of rank 1 matrices. This property is, of course, known, but we do provide a proof for the sake of completeness.

Lemma 2.1. *If $A \in \mathrm{M}_n(\mathcal{K})$, $n \geq 2$, has rank 1, then $A^n B A^{n-1} = A^{n-1} B A^n$ for any matrix $B \in \mathrm{M}_n(\mathcal{K})$.*

Proof. By the Cayley-Hamilton theorem, A satisfies the equation

$$A^n - \mathrm{tr}(A)A^{n-1} = 0,$$

where $\mathrm{tr}(A)$ stands for the trace of the matrix A. Hence

$$A^n B A^{n-1} = \mathrm{tr}(A)A^{n-1}BA^{n-1} = A^{n-1}B\,\mathrm{tr}(A)A^{n-1} = A^{n-1}BA^n,$$

as required. □

Let $\mathrm{L}_n(\mathcal{K})$ denote the semigroup of all $n \times n$-matrices of rank at most 1 over $\mathcal{K}$. From Lemma 2.1 we immediately obtain

Corollary 2.2. *For any field $\mathcal{K}$ and $n \geq 2$, the monoid $\mathrm{L}_n^1(\mathcal{K})$ satisfies the identity*

$$x^n y x^{n-1} = x^{n-1} y x^n. \tag{2.1}$$

Observe that every group satisfying (2.1) is abelian.

2.2. **Matrix semigroups with Moore-Penrose inverse.** We first recall the notion of Moore-Penrose inverse. This has been discovered by Moore [29] and independently by Penrose [34] for complex matrices, but has turned out to be a fruitful concept in a more general setting—see [7] for a comprehensive treatment.

The following results were obtained by Drazin [11].

Proposition 2.3. [11, Proposition 1] *Let $\mathcal{S}$ be an involutory semigroup. Then, for any given $a \in \mathcal{S}$, the four equations*

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa \tag{2.2}$$

have at most one common solution $x \in \mathcal{S}$.

For an element a of an involutory semigroup $\mathcal{S}$, we denote by $a^\dagger$ the unique common solution x of the equations (2.2), provided it exists, and call $a^\dagger$ the *Moore-Penrose inverse* of a.

Recall that an element $a \in \mathcal{S}$ is said to be *regular*, if there is an $x \in \mathcal{S}$ such that $axa = a$. Concerning existence of the Moore-Penrose inverse, we have the following

Proposition 2.4. [11, Proposition 2] *Let $\mathcal{S}$ be an involutory semigroup satisfying the quasi-identity*

$$x^*x = x^*y = y^*x = y^*y \to x = y. \tag{2.3}$$

Then for an arbitrary $a \in \mathcal{S}$, the Moore-Penrose inverse $a^\dagger$ exists if and only if a^*a and aa^* are regular elements.

Let $\langle R, +, \cdot\rangle$ be a ring. An *involution of the ring* is an involution $x \mapsto x^*$ of the semigroup $\langle R, \cdot\rangle$ satisfying in addition the identity $(x+y)^* = x^* + y^*$. For ring involutions, the quasi-identity (2.3) is easily seen to be equivalent to

$$x^*x = 0 \to x = 0. \tag{2.4}$$

Now suppose that $\mathcal{K} = \langle K, +, \cdot\rangle$ is a field that admits an involution $x \mapsto \overline{x}$. Then the matrix ring $\mathrm{M}_n(\mathcal{K})$ has an involution that naturally arises from the involution of $\mathcal{K}$, namely $(a_{ij}) \mapsto (a_{ij})^* := (\overline{a_{ij}})^T$. This involution of $\mathrm{M}_n(\mathcal{K})$ in general does not satisfy the quasi-identity (2.4). However, it does satisfy (2.4) if and only if the equation

$$x_1\overline{x_1} + x_2\overline{x_2} + \cdots + x_n\overline{x_n} = 0 \tag{2.5}$$

admits only the trivial solution $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in K^n. Since all elements of $\mathrm{M}_n(\mathcal{K})$ are regular, this means that the Moore-Penrose inverse exists — subject to the involution $(a_{ij}) \mapsto (a_{ij})^* = (\overline{a_{ij}})^T$ — whenever (2.5) admits only the trivial solution. (The classical Moore-Penrose inverse is thereby obtained by putting $\mathcal{K} = \mathbb{C}$, the field of complex numbers, endowed with the usual complex conjugation $z \mapsto \overline{z}$.) On the other hand, it is easy to see that the condition that (2.5) has only the trivial solution is necessary: if $(a_1, \ldots, a_n)$ were a non-trivial solution to (2.5), then the matrix formed by n identical rows $(a_1, \ldots, a_n)$ would have no Moore-Penrose inverse.

The proof of the main result of this subsection requires an explicit calculation of the Moore-Penrose inverses of certain rank 1 matrices. Thus, we present a simple method for such a calculation. For a row vector $a = (a_1, \ldots, a_n) \in K^n$, where $\mathcal{K} = \langle K, +, \cdot\rangle$ is a field with an involution $x \mapsto \overline{x}$, let a^* denote the column vector $(\overline{a_1}, \ldots, \overline{a_n})^T$. It is easy to see that any $n \times n$-matrix A of rank 1 over $\mathcal{K}$ can be represented as $A = b^*c$ for some non-zero row vectors $b, c \in K^n$. Provided that (2.5) admits only the trivial solution in K^n, one gets $A^\dagger$ as follows:

$$A^\dagger = c^*(cc^*)^{-1}(bb^*)^{-1}b. \tag{2.6}$$

Here bb^* and cc^* are non-zero elements of $\mathcal{K}$ whence their inverses in $\mathcal{K}$ exist. In order to justify (2.6), it suffices to check that the right hand side of (2.6) satisfies the simultaneous equations (2.2) with the matrix A in the role of a, and this is straightforward. Note that formula (2.6) immediately shows that $A^\dagger$ is a scalar multiple of $A^* = c^*b$, namely

$$A^\dagger = \frac{1}{cc^* \cdot bb^*}A^*. \tag{2.7}$$

So, we can formulate one of the highlights of the section — a result that reveals an unexpected feature of a rather classical and well studied object.

Theorem 2.5. *Let $\mathcal{K} = \langle K, +, \cdot\rangle$ be a field having an involution $x \mapsto \overline{x}$ for which the equation $x\overline{x} + y\overline{y} = 0$ has only the trivial solution $(x, y) = (0, 0)$*

in K^2. Then the unary semigroup $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^\dagger \rangle$ of all 2×2-matrices over $\mathcal{K}$ endowed with Moore-Penrose inversion † — subject to the involution $(a_{ij}) \mapsto (a_{ij})^* = (\overline{a_{ij}})^T$ — has no finite basis of identities.

Proof. Set $\mathcal{S} = \langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^\dagger \rangle$. By Theorem 1.2 and Lemma 1.1 it is sufficient to show that

1) $\mathcal{K}_3 \in \mathsf{var}\,\mathcal{S}$,
2) there exists a group $\mathcal{G} \in \mathsf{var}\,\mathcal{S}$ such that $\mathcal{G} \notin \mathsf{var}\,\mathrm{H}(\mathcal{S})$.

In order to prove 1), consider the following sets of rank 1 matrices in $\mathrm{M}_2(\mathcal{K})$:

$$\begin{aligned} H_{11} &= \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \right\}, & H_{12} &= \left\{ \begin{pmatrix} x & 0 \\ x & 0 \end{pmatrix} \right\}, & H_{13} &= \left\{ \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \right\}, \\ H_{21} &= \left\{ \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} \right\}, & H_{22} &= \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\}, & H_{23} &= \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\}, \\ H_{31} &= \left\{ \begin{pmatrix} 0 & 0 \\ x & x \end{pmatrix} \right\}, & H_{32} &= \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right\}, & H_{33} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \right\}, \end{aligned} \tag{2.8}$$

where in each case x runs over $K \setminus \{0\}$. Observe that $\mathcal{K}$ cannot be of characteristic 2, since the equation $x\overline{x} + y\overline{y} = 0$ has only the trivial solution in K^2. Taking this into account, a straightforward calculation shows that

$$H_{ij} \cdot H_{k\ell} = \begin{cases} H_{i\ell} & \text{if } (j,k) \neq (2,3), (3,2), \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Hence the set

$$T = \bigcup_{1 \leq i,j \leq 3} H_{ij} \cup \{0\}$$

is closed under multiplication so that this set forms a subsemigroup $\mathcal{T}$ of $\mathcal{S}$ and the partition $\mathscr{H}$ of T into the classes H_{ij} and $\{0\}$ is a congruence on $\mathcal{T}$. Equation (2.7) shows that

$$H_{ij}^\dagger = H_{ji}. \tag{2.10}$$

We see that T is closed under Moore-Penrose inversion and $\mathscr{H}$ respects †, thus is a congruence on the unary semigroup $\mathcal{T}' = \left\langle T, \cdot, {}^\dagger \right\rangle$. Now comparing (2.9) and (2.10) with the multiplication and inversion rules in $\mathcal{K}_3$ (see (1.1)), we conclude that $\mathcal{T}'/\mathscr{H}$ and $\mathcal{K}_3$ are isomorphic as unary semigroups. Hence $\mathcal{K}_3$ is in $\mathsf{var}\,\mathcal{S}$.

For 2) we merely let $\mathrm{GL}_2(\mathcal{K})$, the group of all invertible 2×2-matrices over $\mathcal{K}$, play the role of $\mathcal{G}$. Since Moore-Penrose inversion on $\mathrm{GL}_2(\mathcal{K})$ coincides with usual matrix inversion, we observe that $\mathrm{GL}_2(\mathcal{K})$ is a unary subsemigroup of $\mathcal{S}$. Moreover, since $AA^\dagger$ is the identity matrix for every invertible matrix A, we conclude that, with the exception of the identity matrix, the Hermitian subsemigroup $\mathrm{H}(\mathcal{S})$ contains only matrices of rank 1, that is, $\mathrm{H}(\mathcal{S}) \subseteq \mathrm{L}_2^1(\mathcal{K})$, the unary semigroup of all matrices of rank at most 1 with the identity matrix adjoined. By Corollary 2.2 the monoid $\mathrm{L}_2^1(\mathcal{K})$ satisfies the identity $x^2yx = xyx^2$. Consequently, each group in $\mathsf{var}\,\mathrm{H}(\mathcal{S})$ is abelian,

while the group $\mathrm{GL}_2(\mathcal{K})$ is non-abelian. Thus, $\mathrm{GL}_2(\mathcal{K})$ is contained in $\mathsf{var}\,\mathcal{S}$ but is not contained in $\mathsf{var}\,\mathrm{H}(\mathcal{S})$, as required. $\square$

Remark 2.1. Apart from any subfield of $\mathbb{C}$ closed under complex conjugation, Theorem 2.5 applies, for instance, to finite fields $\mathcal{K} = \langle K, +, \cdot\rangle$ for which $|K| \equiv 3 \pmod 4$, endowed with the trivial involution $x \mapsto \overline{x} = x$; the latter follows from the fact that the equation $x^2+1=0$ admits no solution in $\mathcal{K}$ if and only if $|K| \equiv 3 \pmod 4$ (cf. [22, Theorem 3.75]). Moreover, by slightly changing the arguments one can show an analogous result for $\mathcal{K}$ being any skew-field of quaternions closed under conjugation.

The reader may ask whether or not the restriction on the size of matrices is essential in Theorem 2.5. For some fields, it definitely is. For instance, for finite fields with the trivial involution $x \mapsto \overline{x} = x$, no extension of Theorem 2.5 to $n \times n$-matrices with $n > 2$ is possible simply because the Moore-Penrose inverse is only a partial operation in this case. Indeed, it is a well known corollary of the Chevalley-Warning theorem (see, e.g., [42, Corollary 2 in §1.2]) that the equation $x_1^2+\cdots+x_n^2=0$ (that is (2.5) with the trivial involution) admits a non-trivial solution in any finite field whenever $n > 2$.

The situation is somewhat more complicated for subfields of $\mathbb{C}$. Theorem 1.2 does not apply here because of the following obstacle. It is well known (see, for example, [23, p. 101]) that the two matrices

$$\zeta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

generate a free subgroup of $\left\langle \mathrm{SL}_2(\mathbb{Z}), \cdot, {}^{-1}\right\rangle$. On the other hand, it is easy to verify that for any subfield $\mathcal{K}$ of $\mathbb{C}$ closed under complex conjugation, the mapping $\langle \mathrm{SL}_2(\mathbb{Z}), \cdot, {}^{-1}\rangle \to \langle \mathrm{M}_3(\mathcal{K}), \cdot, {}^{\dagger}\rangle$ defined by $A \mapsto \left(\begin{smallmatrix} \multicolumn{2}{c}{A} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is an embedding of unary semigroups. Thus, for $n > 2$, the unary semigroup $\langle \mathrm{Sing}_n(\mathcal{K}), \cdot, {}^{\dagger}\rangle$ of singular $n \times n$-matrices contains a free non-abelian group, whence every group belongs to the unary semigroup variety generated by the unary monoid $\langle \mathrm{Sing}_n^1(\mathcal{K}), \cdot, {}^{\dagger}\rangle$. Now we observe that $\mathrm{Sing}_n^1(\mathcal{K})$ is contained in (actually, coincides with) the Hermitian subsemigroup of $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^{\dagger}\rangle$. Indeed, it was proved in [12] (see also [2] for a recent elementary proof) that the semigroup $\langle \mathrm{Sing}_n^1(\mathcal{K}), \cdot\rangle$ is generated by idempotent matrices. For an arbitrary idempotent matrix $A \in \mathrm{M}_n(\mathcal{K})$, let

$$N(A) = \{x \in K^n \mid xA = 0\} \quad \text{and} \quad F(A) = \{x \in K^n \mid xA = x\}$$

be the null-space and the fixed-point-space of A, respectively. Now consider two matrices of orthogonal projectors: P_1, the matrix of the orthogonal projector to the space $F(A)$, and P_2, the matrix of the orthogonal projector to the space $N(A)^{\perp}$. As any orthogonal projector matrix P satisfies $P = P^2 = P^{\dagger}$, both $P_1 = P_1P_1^{\dagger}$ and $P_2 = P_2P_2^{\dagger}$ belong to the Hermitian subsemigroup $\mathrm{H}(\mathrm{M}_n(\mathcal{K}))$, but then A also belongs to $\mathrm{H}(\mathrm{M}_n(\mathcal{K}))$ since

$A = (P_1P_2)^\dagger$, see [28, Exercise 5.15.9a]. Thus, $\mathrm{Sing}_n^1(\mathcal{K}) \subseteq \mathrm{H}(\mathrm{M}_n(\mathcal{K}))$, whence no group $\mathcal{G}$ can satisfy the condition of Theorem 1.2.

However, the fact that Theorem 1.2 cannot be applied to, say, the unary semigroup $\langle \mathrm{M}_3(\mathbb{C}), \cdot, {}^\dagger\rangle$ does not yet mean that the identities of this semigroup are finitely based. We thus have the following open question.

Problem 2.1. Is the unary semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^\dagger\rangle$ not finitely based for each subfield $\mathcal{K}$ of $\mathbb{C}$ closed under complex conjugation and for all $n > 2$?

In connection with this problem, we observe that the proofs of Theorem 2.5 and Corollary 2.2 readily yield the following:

Remark 2.2. For each conjugation-closed subfield $\mathcal{K}$ of $\mathbb{C}$ and for all $n > 2$, the unary semigroup $\langle \mathrm{L}_n(\mathcal{K}) \cup \mathcal{G}, \cdot, {}^\dagger\rangle$ consisting of all matrices of rank at most 1 and all matrices from some non-abelian subgroup $\mathcal{G}$ of $\mathrm{GL}_n(\mathcal{K})$ has no finite identity basis.

Another natural related structure is the semigroup $\mathrm{M}_n(\mathcal{K})$ endowed with **both** unary operations † and *. Here our techniques produce a similar result.

Theorem 2.6. *Let $\mathcal{K}$ be a field as in Theorem 2.5; then $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^\dagger, {}^*\rangle$ is not finitely based as an algebraic structure of type* $(2, 1, 1)$.

Proof. The characteristic of $\mathcal{K}$ is not 2 whence the group

$$\mathcal{G} = \{A \in \mathrm{GL}_2(\mathcal{K}) \mid A^\dagger = A^*\}$$

is non-abelian. Indeed, on the prime subfield of $\mathcal{K}$, the involution $x \mapsto \overline{x}$ is the identity automorphism; so, for matrices over the prime subfield, conjugation * coincides with transposition, and thus, for example, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are two non-commuting members of $\mathcal{G}$. Set $\mathcal{A} = \langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^\dagger, {}^*\rangle$; the algebraic structure $\langle G, \cdot, {}^\dagger, {}^*\rangle$, that is, the group $\mathcal{G}$ with inversion taken twice as unary operation, belongs to $\mathsf{var}\,\mathcal{A}$. Now, as in the proof of Theorem 2.5, consider the set

$$T = \bigcup_{1 \le i,j \le 3} H_{ij} \cup \{0\}$$

where H_{ij} are defined via (2.8). Obviously, $H_{ij}^* = H_{ji}$ whence $\mathcal{T} = \langle T, \cdot, {}^\dagger, {}^*\rangle$ is a substructure of $\mathcal{A}$ and the partition $\mathscr{H}$ of T into the classes H_{ij} and $\{0\}$ is a congruence on this substructure. The quotient $\mathcal{T}/\mathscr{H}$ is then isomorphic to the semigroup $\mathcal{K}_3$ endowed twice with its unary operation. We conclude that $\mathcal{K}_3$ treated this way also belongs to $\mathsf{var}\,\mathcal{A}$. By Corollary 2.2 the identity $x^2yx = xyx^2$ holds in $\mathrm{H}(\mathcal{A})$ (by which we mean the substructure of $\mathcal{A}$ generated by all elements of the form $AA^\dagger$). Now construct the semigroups $\mathcal{T}_k$ (by use of the identity $x^2yx = xyx^2$) as in Step 2 in the proof of Theorem 1.2 and endow each of them twice with its unary operation. The arguments in Steps 3 and 4 in the proof then show that $\mathcal{T}_k$ does not belong to $\mathsf{var}\,\mathcal{A}$ while each k-generated substructure of $\mathcal{T}_k$ does belong to $\mathsf{var}\,\mathcal{A}$. Thus, $\mathcal{T}_k$ can play the role of critical structures for $\mathsf{var}\,\mathcal{A}$ whence the desired conclusion follows by the reasoning as in Step 1 in the proof of Theorem 1.2. $\square$

Also in this setting, our result gives rise to a natural question.

Problem 2.2. Is the algebraic structure $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^\dagger, ^* \rangle$ of type $(2,1,1)$ not finitely based for each subfield $\mathcal{K}$ of $\mathbb{C}$ closed under complex conjugation and for all $n > 2$?

Here an observation similar to Remark 2.2 can be stated: for each conjugation-closed subfield $\mathcal{K} \subseteq \mathbb{C}$ and for all $n > 2$, the algebraic structure $\langle \mathrm{L}_n(\mathcal{K}) \cup \mathrm{GL}_n(\mathcal{K}), \cdot, ^\dagger, ^* \rangle$ consisting of all matrices of rank at most 1 and all invertible matrices has no finite identity basis.

2.3. **Matrix semigroups with transposition.** Transposition is certainly the most common unary operation for matrices. For each field $\mathcal{K}$ of characteristic 0, the involutory semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^T \rangle$ is finitely based. Indeed, we have already mentioned that the two matrices ζ and η in (2.11) generate a free subgroup of $\langle \mathrm{SL}_2(\mathbb{Z}), \cdot, ^{-1} \rangle$ and hence a free subsemigroup of $\langle \mathrm{SL}_2(\mathbb{Z}), \cdot \rangle$. But $\eta = \zeta^T$ whence the unary subsemigroup in $\langle \mathrm{SL}_2(\mathbb{Z}), \cdot, ^T \rangle$ generated by ζ is isomorphic to the free monogenic involutory semigroup $\mathrm{FI}(\{\zeta\})$. The latter semigroup is known to contain as a unary subsemigroup a free involutory semigroup on countably many generators, namely, $\mathrm{FI}(Z)$ where

$$Z = \{\zeta\zeta^T\zeta,\ \zeta(\zeta^T)^2\zeta,\ \dots,\ \zeta(\zeta^T)^n\zeta,\ \dots\}.$$

Hence all identities holding in $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^T \rangle$ with $n \geq 2$ and $\mathcal{K}$ of characteristic 0 follow from the associativity and the involution laws $(xy)^T = y^T x^T$, $(x^T)^T = x$. Similarly, $\langle \mathrm{M}_n(R), \cdot, ^* \rangle$ is finitely based for each $n \geq 2$ and each subring $R \subseteq \mathbb{C}$ closed under complex conjugation—here * stands for the complex-conjugate transpostion $(a_{ij})^* = (\overline{a_{ij}})^T$.

We note that Theorem 1.2 and Corollary 2.2 prove the non-existence of a finite identity bases for the unary subsemigroup of $\langle \mathrm{M}_n(\mathbb{C}), \cdot, ^* \rangle$ [respectively $\langle \mathrm{M}_n(\mathbb{R}), \cdot, ^T \rangle$] that consists of all matrices of rank at most 1 together with all unitary [respectively all orthogonal] matrices.

For the case of finite fields, one can show that Theorem 1.2 solves the finite basis problem in the negative for the involutory semigroup $\langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$ for each finite field $\mathcal{K}$ except $\mathcal{K} = \mathbb{F}_2$, the 2-element field. Indeed, it can be verified that the involutory semigroup $\langle \mathrm{M}_2(\mathbb{F}_2), \cdot, ^T \rangle$ satisfies the identity

$$(xx^*)^3(yy^*)^3 = (yy^*)^3(xx^*)^3$$

which does not hold in $\mathcal{K}_3$; consequently, $\mathcal{K}_3$ is not in $\mathsf{var}\langle \mathrm{M}_2(\mathbb{F}_2), \cdot, ^T \rangle$ and Theorem 1.2 does not apply here. In the following theorem, we shall demonstrate the application of Theorem 1.2 only in the case when $\mathcal{K}$ has odd characteristic. With some additional effort we could include also the case when the characteristic of $\mathcal{K}$ is 2 and $|K| \geq 4$. We shall omit this since that case will be covered by a different kind of proof later.

Theorem 2.7. *For each finite field $\mathcal{K}$ of odd characteristic, the involutory semigroup $\langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$ has no finite identity basis.*

Proof. Let $\mathcal{S} = \langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$. As in the proof of Theorem 2.5 one shows that $\mathcal{K}_3$ is in $\mathsf{var}\,\mathcal{S}$. Furthermore, let d be the exponent of the group $\mathrm{GL}_2(\mathcal{K})$. By Corollary 2.2, each group in $\mathrm{P}_d(\mathsf{var}\,\mathcal{S}) = \mathsf{var}\,\mathrm{P}_d(\mathcal{S})$ satisfies the law $x^2yx = xyx^2$ and therefore is abelian. On the other hand, as in the proof of Theorem 2.6, the group $\mathcal{G} = \{A \in \mathrm{GL}_2(\mathcal{K}) \mid A^T = A^{-1}\}$ is in $\mathsf{var}\,\mathcal{S}$ but is non-abelian. Thus, Theorem 1.2 applies. $\square$

The next theorem contains the even characteristic case and proves, in fact, a stronger assertion.

Theorem 2.8. *For each finite field $\mathcal{K} = \langle K, +, \cdot \rangle$ with $|K| \not\equiv 3 \pmod 4$, the involutory semigroup $\langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$ is inherently non-finitely based.*

Proof. As mentioned in Remark 2.1, there exists $x \in K$ for which $1+x^2 = 0$. Now consider the following matrices:

$$H_{11} = \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix}, \; H_{12} = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}, \; H_{21} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \; H_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the set $M = \{H_{11}, H_{12}, H_{21}, H_{22}, I, O\}$ is closed under multiplication and transposition, hence $\mathcal{M} = \langle M, \cdot, ^T \rangle$ is an involutory subsemigroup of $\langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$. The mapping $\mathcal{TA}_2^1 \to \mathcal{M}$ given by

$$(i,j) \mapsto H_{ij}, \; 0 \mapsto O, \; 1 \mapsto I$$

is an isomorphism of involutory semigroups. The result now follows from Corollary 1.7. $\square$

The case of matrix semigroups of degree greater than 2 is similar.

Theorem 2.9. *For each $n \geq 3$ and each finite field $\mathcal{K}$, the involutory semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^T \rangle$ is inherently non-finitely based.*

Proof. It follows from the Chevalley-Warning theorem [42, Corollary 2 in §1.2] that there exist $x, y \in K$ satisfying $1 + x^2 + y^2 = 0$. Now consider the following matrices:

$$H_{11} = \begin{pmatrix} 1 & x & y \\ x & x^2 & xy \\ y & xy & y^2 \end{pmatrix}, \; H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \; H_{21} = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$H_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, the set $M = \{H_{11}, H_{12}, H_{21}, H_{22}, I, O\}$ is closed under multiplication and transposition, and as in the previous proof, $\mathcal{M} = \langle M, \cdot, ^T \rangle$ forms an involutory subsemigroup of $\langle \mathrm{M}_3(\mathcal{K}), \cdot, ^T \rangle$ that is isomorphic with $\mathcal{TA}_2^1$. Hence $\langle \mathrm{M}_3(\mathcal{K}), \cdot, ^T \rangle$ is inherently non-finitely based. The assertion for $\mathrm{M}_n(\mathcal{K})$ for $n \geq 3$ now follows in an obvious way. $\square$

Remark 2.3. The statements of Theorems 2.7, 2.8, 2.9 remain valid if the unary operation $A \mapsto A^T$ is replaced with an operation of the form $A \mapsto A^{\sigma T}$ for any automorphism σ of $\mathcal{K}$, where $(a_{ij})^{\sigma T} := (a_{ij}^{\sigma})^T$.

We are ready to formulate another highlight of this section. With the exception of the 'only if' part of item (2), this is a summary of Theorems 2.7, 2.8, and 2.9.

Theorem 2.10. *Let $n \geq 2$ and $\mathcal{K} = \langle K, +, \cdot \rangle$ be a finite field. Then*

1. *the involutory semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^T \rangle$ is not finitely based;*
2. *the involutory semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, ^T \rangle$ is inherently non-finitely based if and only if either $n \geq 3$ or $n = 2$ and $|K| \not\equiv 3 \pmod 4$.*

It remains to prove that $\langle \mathrm{M}_2(\mathcal{K}), \cdot, ^T \rangle$ is **not** inherently non-finitely based if $|K| \equiv 3 \pmod 4$. The assertion follows from the next proposition.

Proposition 2.11. *Let $\mathcal{S} = \langle S, \cdot, ^* \rangle$ be a finite involutory semigroup and suppose that there exists an involutory word $\omega(x)$ in one variable x such that $\mathcal{S}$ satisfies the identity $x = x\omega(x)x$. Then $\mathcal{S}$ is not inherently non-finitely based.*

Proof. This is a consequence of some deep facts obtained by Margolis and Sapir in [25] and we shall follow their main arguments. The reason that these arguments apply here is that in $\mathcal{S}$, the Green relation $\mathscr{R}$ can be expressed in terms of equational logic. This ensures that we can construct a finite set of identities that defines a locally finite variety of involutory semigroups containing $\mathcal{S}$.

Indeed, since $\mathcal{S}$ satisfies the identity $x = x\omega(x)x$, we have $a \mathrel{\mathscr{R}} a\omega(a)$ for each element $a \in S$. Thus, for $a, b \in S$, we have $a \mathrel{\mathscr{R}} b$ if and only if $a\omega(a) \mathrel{\mathscr{R}} b\omega(b)$. Since $a\omega(a)$ and $b\omega(b)$ are idempotents, that latter condition is equivalent to the two equalities $a\omega(a) \cdot b\omega(b) = b\omega(b)$ and $b\omega(b) \cdot a\omega(a) = a\omega(a)$. In particular, for $u, v \in S$ we have $uv \mathrel{\mathscr{R}} u$ if and only if $uv\omega(uv) \cdot u\omega(u) = u\omega(u)$ (since the second equality $u\omega(u) \cdot uv\omega(uv) = uv\omega(uv)$ is always true).

Recall the definition of the Zimin words Z_n given in Subsection 1.3: $Z_1 = x_1$, $Z_{n+1} = Z_n x_{n+1} Z_n$. Let Z_n' be the word obtained from Z_n by deleting the last letter (which is x_1), that is, $Z_n' x_1 = Z_n$. Further let h denote the *$\mathscr{R}$-height* of $\mathcal{S}$, that is, h is the length of the longest possible $\mathscr{R}$-chain

$$s_1 <_{\mathscr{R}} s_2 <_{\mathscr{R}} \cdots <_{\mathscr{R}} s_k$$

where $a <_{\mathscr{R}} b$ means that $aS^1 \subseteq bS^1$ but $bS^1 \nsubseteq aS^1$. Set $n = h+1$; Lemma 7 in [25] shows that $\mathcal{S}$ satisfies $Z_n' \mathrel{\mathscr{R}} Z_n$ whence it satisfies the identity

$$Z_n\omega(Z_n) \cdot Z_n'\omega(Z_n') = Z_n'\omega(Z_n'). \tag{2.12}$$

On the other hand, each involutory semigroup $\mathcal{T}$ which satisfies $x = x\omega(x)x$ and (2.12) necessarily satisfies $Z_n' \mathrel{\mathscr{R}} Z_n$, hence (its semigroup reduct) belongs to the quasivariety $\mathbf{Q}_n$ of semigroups defined by the quasi-identity

$$xZ_n = yZ_n \to xZ_n' = yZ_n'.$$

Lemma 8 in [25] then shows that a finitely generated semigroup $\mathcal{T} \in \mathbf{Q}_n$ is finite if and only if it is periodic and all subgroups of $\mathcal{T}$ are locally finite. We note that an involutory semigroup is finitely generated if and only if so is its semigroup reduct. Hence it suffices to find a finite number of identities which hold in $\mathcal{S}$ and which force each (involutory) semigroup to be periodic and to have only locally finite subgroups.

We can proceed as at the end of [25]: first, since $\mathcal{S}$ is finite, it satisfies the identity $x^k = x^{k+\ell}$ for some $k, \ell \geq 1$. This identity definitely forces any (involutory) semigroup to be periodic. Next, let $\mathcal{G}$ be the direct product of all maximal subgroups of $\mathcal{S}$. By the Oates-Powell theorem [33], see also [32, §5.2], the locally finite variety $\mathsf{var}\,\mathcal{G}$ generated by the finite group $\mathcal{G}$ can be defined by a single identity $v(x_1, \ldots, x_m) = 1$. The left hand side v of this identity can be assumed to contain no negative letters, that is, v is a plain semigroup word in the letters $x_1, \ldots, x_m$. Now let $\mathcal{F} = \mathcal{F}(x_1, \ldots, x_m)$ be the m-generated relatively free semigroup in the (locally finite) semigroup variety generated by the semigroup reduct of $\mathcal{S}$. Let e be an idempotent in the minimal ideal of $\mathcal{F}$ and let $u(x_1, \ldots, x_m)$ be a word whose value in $\mathcal{F}$ is e. It follows that $\mathcal{F}$ (and therefore $\mathcal{S}$) satisfies the identity

$$u = u^2. \tag{2.13}$$

For every element $g \in \mathcal{F}$, the product ege belongs to the maximal subgroup of $\mathcal{F}$ with idempotent e. Consequently, $\mathcal{F}$ (and therefore $\mathcal{S}$) satisfies the identity

$$v(ux_1u, \ldots, ux_mu) = u. \tag{2.14}$$

Note that both sides of that identity are plain semigroup words in the letters $x_1, \ldots, x_m$.

Now consider the variety of involutory semigroups defined by the identities $x = x\omega(x)x$, $x^k = x^{k+\ell}$ together with (2.12), (2.13) and (2.14). By construction, $\mathcal{S}$ is a member of that variety. Let $\mathcal{T}$ be any finitely generated member; then, as already mentioned, the semigroup reduct of $\mathcal{T}$ is also finitely generated. The first and the third identity ensure that the semigroup reduct of $\mathcal{T}$ belongs to the quasivariety $\mathbf{Q}_n$, and therefore it is finite provided that it is periodic and all its subgroups are locally finite. Periodicity is, of course, guaranteed by the second identity. Finally, each group $\mathcal{H}$ that satisfies the identities (2.13) and (2.14) satisfies the identity $v(x_1, \ldots, x_m) = 1$, whence $\mathcal{H}$ belongs to $\mathsf{var}\,\mathcal{G}$ and so is locally finite. Altogether, $\mathcal{T}$ is finite and the proposition is proved. □

Remark 2.4. Proposition 2.11 implies in particular that no finite regular $*$-semigroup can be inherently non-finitely based as one can use x^* in the role of the term $\omega(x)$. A stronger result has been established by the second author [10].

Proof of Theorem 2.10. Suppose that $|K| \equiv 3 \pmod 4$. This is precisely the case when each matrix A in $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^T \rangle$ admits a Moore-Penrose inverse $A^\dagger$ (Remark 2.1). Let A be a matrix of rank 1; by (2.7) there exists a scalar

$\alpha \in \mathcal{K} \setminus \{0\}$ such that $\alpha A^\dagger = A^T$. Let $r = |K| - 1$; then $\alpha^r = 1$. Since the multiplicative subgroup of $\mathcal{K}$ is a cyclic subgroup of $\mathrm{GL}_2(\mathcal{K})$, the number r divides the exponent d of $\mathrm{GL}_2(\mathcal{K})$ whence $\alpha^d = 1$. Consequently,

$$A(A^T A)^d = A(\alpha A^\dagger A)^d = \alpha^d A(A^\dagger A)^d = A.$$

If $A \in \mathrm{GL}_2(\mathcal{K})$, we also have $A = A(A^T A)^d$ because $(A^T A)^d$ is the identity matrix; clearly, the equality $A = A(A^T A)^d$ holds also for the case when A is the zero matrix. Summarizing, we conclude that the identity $x = x(x^T x)^d$ holds in the involutory semigroup $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^T\rangle$. Setting $\omega(x) := x^T(xx^T)^{d-1}$, we see that $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^T\rangle$ satisfies the identity $x = x\omega(x)x$, as required by Proposition 2.11. □

Remark 2.5. It is known [38, Corollary 6.2] that the matrix semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot\rangle$ is inherently non-finitely based (as a plain semigroup) for every $n \geq 2$ and every finite field $\mathcal{K}$. Thus, the involutory semigroups $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^T\rangle$ over finite fields $\mathcal{K}$ such that $|K| \equiv 3 \pmod 4$ provide a natural series of unary semigroups whose equational properties essentially differ from the equational properties of their semigroup reducts.

2.4. **Matrix semigroups with symplectic transpose.** For a $2n \times 2n$-matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D being $n \times n$-matrices over any field $\mathcal{K}$, the *symplectic transpose* X^S is defined by

$$X^S = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},$$

see e.g. [36, (5.1.1)]. This definition is similar to that of the involution in the twisted Brandt monoid $\mathcal{TB}_2^1$ (defined in terms of 2×2-matrices) and leads to the following application.

Theorem 2.12. *The involutory semigroup $\langle \mathrm{M}_{2n}(\mathcal{K}), \cdot, {}^S\rangle$ is inherently non-finitely based for each $n \geq 1$ and each finite field $\mathcal{K} = \langle K, +, \cdot\rangle$.*

Proof. Consider the following sets of $2n \times 2n$-matrices:

$$H_{11} = \left\{ \pm \begin{pmatrix} O_n & I_n \\ O_n & O_n \end{pmatrix} \right\},\ H_{12} = \left\{ \pm \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix} \right\},$$

$$H_{21} = \left\{ \pm \begin{pmatrix} O_n & O_n \\ O_n & I_n \end{pmatrix} \right\},\ H_{22} = \left\{ \pm \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix} \right\}$$

where for any positive integer k, we denote be I_k, respectively, O_k the identity, respectively, zero $k \times k$-matrix. Let

$$T = \bigcup_{1 \leq i,j \leq 2} H_{ij} \cup \{O_{2n}, I_{2n}\}.$$

The set T is closed under multiplication and symplectic transposition whence $\mathcal{T} = \left\langle T, \cdot, {}^S \right\rangle$ forms an involutory subsemigroup of $\left\langle \mathrm{M}_{2n}(\mathcal{K}), \cdot, {}^S \right\rangle$. On the other hand, the mapping

$$H_{ij} \mapsto (i,j),\ I_{2n} \mapsto 1,\ O_{2n} \mapsto 0$$

is a homomorphism of $\mathcal{T}$ onto $\mathcal{TB}_2^1$. Altogether, the twisted Brandt monoid $\mathcal{TB}_2^1$ divides $\left\langle \mathrm{M}_{2n}(\mathcal{K}), \cdot, {}^S \right\rangle$. □

2.5. **Boolean matrices.** Recall that a *Boolean* matrix is a matrix with entries 0 and 1 only. The multiplication of such matrices is as usual, except that addition and multiplication of the entries is defined as: $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$. Let B_n denote the set of all Boolean $n \times n$-matrices. It is well known that the semigroup $\langle B_n, \cdot \rangle$ is essentially the same as the semigroup of all binary relations on an n-element set subject to the usual composition of binary relations. The operation ^{T} of forming the matrix transpose then corresponds to the operation of forming the dual binary relation.

Theorem 2.13. *For each integer $n \geq 2$, the involutory semigroup $\mathcal{B}_n = \langle B_n, \cdot, {}^T \rangle$ of all Boolean $n \times n$-matrices endowed with transposition is inherently non-finitely based.*

Proof. Consider the Boolean matrices

$$B_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ B_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ B_{21} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ B_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$
$$O = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set $M = \{B_{11}, B_{12}, B_{21}, B_{22}, O, I\}$ is closed under multiplication and transposition whence $\mathcal{M} = \langle M, \cdot, {}^T \rangle$ is an involutory subsemigroup of $\mathcal{B}_2$. The mapping $\mathcal{TB}_2^1 \to \mathcal{M}$ given by

$$(i,j) \mapsto B_{ij},\ 0 \mapsto O,\ 1 \mapsto I$$

is an isomorphism of involutory semigroups. By Corollary 1.6 $\mathcal{M}$ is inherently non-finitely based whence so is $\mathcal{B}_2$. Since $\mathcal{B}_2$ can be embedded as an involutory semigroup into $\mathcal{B}_n$ for each n, the result follows. □

Remark 2.6. We can unify Theorem 2.13 and some results in Subsection 2.3 by considering matrix semigroups with transposition over **semirings**. A *semiring* is an algebraic structure $\mathcal{L} = \langle L, +, \cdot \rangle$ of type $(2,2)$ such that $\langle L, + \rangle$ is a commutative semigroup, $\langle L, \cdot \rangle$ is a semigroup and multiplication distributes over addition. From the proofs of Theorems 2.8 and 2.13 we see that the involutory matrix semigroup $\langle \mathrm{M}_n(\mathcal{L}), \cdot, {}^T \rangle$ over a finite semiring is inherently non-finitely based whenever $n \geq 2$ and the semiring $\mathcal{L}$ has a zero 0 (that is, a neutral element for $\langle L, + \rangle$ which is at the same time an absorbing element for $\langle L, \cdot \rangle$) and satisfies either of the following two conditions:

(1) there exist (not necessarily distinct) elements $e, x \neq 0$ such that $e^2 = e$, $ex = xe = x$, $e + x^2 = 0$;
(2) there exists an element $e \neq 0$ such that $e^2 = e = e + e$.

We have already met an infinite series of semirings satisfying (1): it consists of the finite fields $\mathcal{K} = \langle K, +, \cdot \rangle$ with $|K| \not\equiv 3 \pmod 4$. It should be noted that semirings satisfying (2) are even more plentiful: for example, finite distributive lattices as well as the power semirings of finite semigroups (with the subset union as addition and the subset product as multiplication) fall in this class.

3. Applications to partition semigroups

3.1. **Partition semigroups.** For each positive integer n we are going to define

- the *partition monoid* $\mathfrak{C}_n$,
- the *Brauer monoid* $\mathfrak{B}_n$,
- the *partial* Brauer monoid $P\mathfrak{B}_n$,
- the *annular monoid* $\mathfrak{A}_n$,
- the *partial* annular monoid $P\mathfrak{A}_n$.

The monoids $\mathfrak{C}_n$, $\mathfrak{B}_n$ and $\mathfrak{A}_n$ arise as vector space bases of certain associative algebras which are relevant in representation theory [6,13,16,45]. The semigroup structure and related questions for $\mathfrak{C}_n$, $P\mathfrak{B}_n$ and $\mathfrak{B}_n$ have been studied recently by Mazorchuk et al., see, for example, [20,21,24,26,27]. For each n there are natural monoid embeddings

$$\mathfrak{A}_n \hookrightarrow \mathfrak{B}_n \hookrightarrow P\mathfrak{B}_n \hookrightarrow \mathfrak{C}_n.$$

Moreover, for each n there are monoid embeddings

$$\mathfrak{C}_n \hookrightarrow \mathfrak{C}_{n+1}, \quad P\mathfrak{B}_n \hookrightarrow P\mathfrak{B}_{n+1}, \quad \mathfrak{B}_n \hookrightarrow \mathfrak{B}_{n+1}.$$

For the annular monoid there is no obvious monoid embedding

$$\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+k}$$

for any $n \geq 2, k \geq 1$, but $\mathfrak{A}_n$ appears as a sub*semigroup* of $\mathfrak{A}_{n+2}$ for each n.

We start with the definition of $\mathfrak{C}_n$. For each positive integer n let

$$[n] = \{1, \ldots, n\}, \quad [n]' = \{1', \ldots, n'\}, \quad [n]'' = \{1'', \ldots, n''\}$$

be three pairwise disjoint copies of the set of the first n positive integers and put

$$\widetilde{[n]} = [n] \cup [n]'.$$

The base set of the partition monoid $\mathfrak{C}_n$ is the set of all partitions of the set $\widetilde{[n]}$; throughout, we consider a partition of a set and the corresponding equivalence relation on that set as two different views of the same thing and without further mention we freely switch between these views, whenever it seems to be convenient. For $\xi, \eta \in \mathfrak{C}_n$, the product $\xi\eta$ is defined (and computed) in four steps:

(1) Consider the $'$-analogue of η: that is, define η' on $[n]' \cup [n]''$ by

$$x' \; \eta' \; y' :\Leftrightarrow x \; \eta \; y \text{ for all } x, y \in \widetilde{[n]}.$$

(2) Let $\langle \xi, \eta \rangle$ be the equivalence relation on $\widetilde{[n]} \cup [n]''$ generated by $\xi \cup \eta'$, that is, set $\langle \xi, \eta \rangle := (\xi \cup \eta')^t$ where ^{t} denotes the transitive closure.

(3) Forget all elements having a single prime $'$: that is, set

$$\langle \xi, \eta \rangle^\circ := \langle \xi, \eta \rangle \,|_{[n] \cup [n]''}.$$

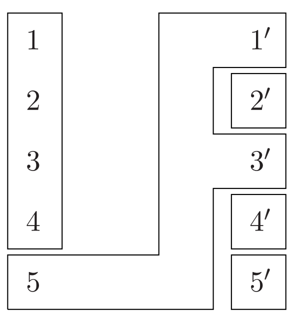
(4) Replace double primes with single primes to obtain the product $\xi\eta$: that is, set

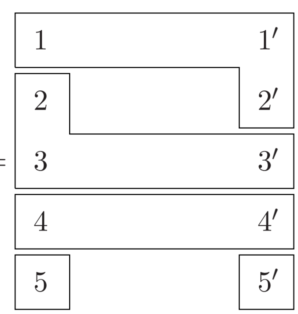
$$x \; \xi\eta \; y :\Leftrightarrow f(x) \; \langle \xi, \eta \rangle^\circ \; f(y) \text{ for all } x, y \in \widetilde{[n]}$$

where $f : \widetilde{[n]} \to [n] \cup [n]''$ is the bijection

$$x \mapsto x, x' \mapsto x'' \text{ for all } x \in [n].$$

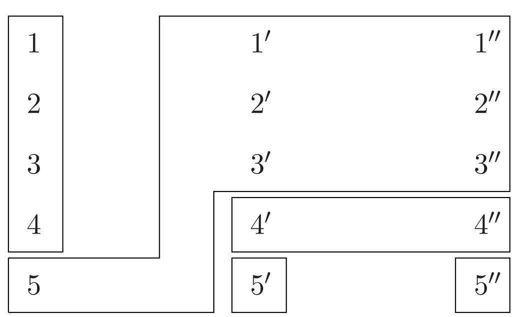
For example, let $n = 5$ and

$\xi =$ 

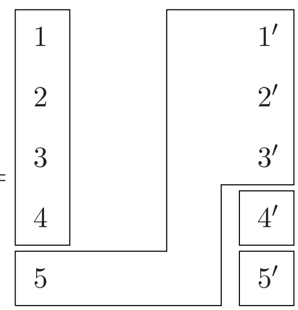
, $\eta =$ 

.

Then

$\langle \xi, \eta \rangle =$ 

and

$\xi\eta =$  .

This multiplication is associative making $\mathfrak{C}_n$ a monoid with identity 1 where

$$1 = \{\{k, k'\} \mid k \in [n]\}.$$

The group of units of $\mathfrak{C}_n$ is the symmetric group $\mathfrak{S}_n$ (acting on $[n]$ on the right) with canonical embedding $\mathfrak{S}_n \hookrightarrow \mathfrak{C}_n$ given by

$$\sigma \mapsto \{\{k, (k\sigma)'\} \mid k \in [n]\} \text{ for all } \sigma \in \mathfrak{S}_n.$$

More generally, the monoid of all (partial) transformations of $[n]$ acting on the right is also naturally embedded in $\mathfrak{C}_n$ by

$$\phi \mapsto \{\{k'\} \cup k\phi^{-1} \mid k \in [n]\} \cup \{\{k\} \mid k \in [n], k \notin \operatorname{dom}\phi\} \tag{3.1}$$

where $\operatorname{dom}\phi$ is the domain of ϕ. The equivalence classes of some $\xi \in \mathfrak{C}_n$ are usually referred to as *blocks*; the *rank* $\operatorname{rk}\xi$ is the number of blocks of ξ whose intersection with $[n]$ as well as with $[n]'$ is not empty — this coincides with the usual notion of rank of a partial mapping on $[n]$ in case ξ is in the image of the embedding (3.1). It is known that the rank characterizes the $\mathscr{D}$-relation in $\mathfrak{C}_n$ [21, 26]: for any $\xi, \eta \in \mathfrak{C}_n$, one has $\xi \mathrel{\mathscr{D}} \eta$ if and only if $\operatorname{rk}\xi = \operatorname{rk}\eta$.

The monoid $\mathfrak{C}_n$ admits a natural involution making it a regular $*$-semigroup: consider first the permutation * on $\widetilde{[n]}$ that swaps primed with unprimed elements, that is, set

$$k^* = k',\ (k')^* = k \text{ for all } k \in [n].$$

Then define, for $\xi \in \mathfrak{C}_n$,

$$x\ \xi^*\ y :\Leftrightarrow x^*\ \xi\ y^* \text{ for all } x, y \in \widetilde{[n]}.$$

That is, ξ^* is obtained from ξ by interchanging in ξ the primed with the unprimed elements. It is easy to see that

$$\xi^{**} = \xi,\ (\xi\eta)^* = \eta^*\xi^* \text{ and } \xi\xi^*\xi = \xi \text{ for all } \xi, \eta \in \mathfrak{C}_n. \tag{3.2}$$

The elements of the form $\xi\xi^*$ are called *projections*. They are idempotents (as one readily sees from the last equality in (3.2)) and have the following transparent structure. If k is the rank of $\xi\xi^*$ (equal to the rank of ξ), then there is some $t \in \{0, 1, \dots, n-k\}$ and a partition of $[n]$ into $k + t$ blocks:

$$[n] = A_1 \cup \cdots \cup A_k \cup B_1 \cup \cdots \cup B_t$$

such that

$$\xi\xi^* = \{A_1 \cup A_1', \dots, A_k \cup A_k', B_1, B_1', \dots, B_t, B_t'\}.$$

Fig. 2 shows a typical projection in $\mathfrak{C}_8$; here $k = 3$, $t = 2$ and $A_1 = \{3, 4\}$, $A_2 = \{5, 8\}$, $A_3 = \{7\}$ while $B_1 = \{1, 2\}$, $B_2 = \{6\}$.

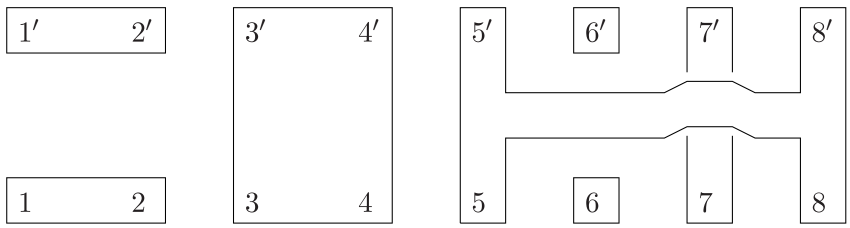


FIGURE 2. A rank 3 projection in $\mathfrak{C}_8$

From this it follows easily that the maximal subgroup of $\mathfrak{C}_n$ with identity $\xi\xi^*$ is isomorphic to the symmetric group $\mathfrak{S}_k$ where the isomorphism between $\mathfrak{S}_k$ and the group $\mathscr{H}$-class of $\xi\xi^*$ is given by

$$\sigma \mapsto \{A_1 \cup A_{1\sigma}', \dots A_k \cup A_{k\sigma}', B_1, B_1', \dots B_t, B_t'\},\ \sigma \in \mathfrak{S}_k. \tag{3.3}$$

We note that in the group $\mathscr{H}$-class of any projection, the involution * coincides with the inverse operation in that group. Since $\xi \mathrel{\mathscr{R}} \xi\xi^*$, *each* maximal subgroup of $\mathfrak{C}_n$ is isomorphic to the symmetric group $\mathfrak{S}_k$ for some $k \leq n$.

The *Brauer monoid* and the *partial* Brauer monoid can be conveniently defined as submonoids of $\mathfrak{C}_n$: namely, $\mathfrak{B}_n$ [respectively $P\mathfrak{B}_n$] consists of all elements of $\mathfrak{C}_n$ all of whose blocks have size 2 [at most 2]. Both monoids are closed under the involution * and in both cases, the group $\mathscr{H}$-class of a projection of rank k is isomorphic (as a regular $*$-semigroup) with the symmetric group $\mathfrak{S}_k$. Now let $\mathfrak{K}_n$ be any of $\mathfrak{C}_n$, $P\mathfrak{B}_n$ or $\mathfrak{B}_n$. Each projection different from the identity of $\mathfrak{K}_n$ has rank less than n, whence the monoid $\mathrm{H}(\mathfrak{K}_n)$ contains, apart from the identity element, only elements of rank strictly less than n. This implies

Corollary 3.1. *For each $n \geq 2$ and each $\mathfrak{K}_n \in \{\mathfrak{C}_n, P\mathfrak{B}_n, \mathfrak{B}_n\}$ there exists a group in $\mathsf{var}\,\mathfrak{K}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$.*

Proof. The group in question is the symmetric group $\mathfrak{S}_n$ that is the group of units in $\mathfrak{K}_n$ and thus belongs to $\mathsf{var}\,\mathfrak{K}_n$. By the argument of Kim and Roush [18], each group in the variety $\mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$ belongs to the group variety generated by the subgroups of the monoid $\mathrm{H}(\mathfrak{K}_n)$. As observed above, each subgroup of $\mathrm{H}(\mathfrak{K}_n)$ embeds into the symmetric group $\mathfrak{S}_{n-1}$ whence it remains to check that $\mathfrak{S}_n$ does not belong to the group variety generated by $\mathfrak{S}_{n-1}$. This follows from [32, Theorem 51.2] because the group $\mathfrak{S}_n$ always has a chief factor of order larger than the maximum order of chief factors in $\mathfrak{S}_{n-1}$. □

For completeness we note that $\mathfrak{B}_1$ is the trivial monoid and $P\mathfrak{B}_1 \cong \mathfrak{C}_1$ is isomorphic to the 2-element semilattice monoid $\{0, 1\}$ (endowed with trivial involution).

Next we define the *annular monoid* $\mathfrak{A}_n$ [16]. It will be realized as a certain submonoid of the Brauer monoid. For this purpose it is convenient to first represent the elements of $\mathfrak{B}_n$ as *annular diagrams*. Consider an annulus A in the complex plane, say $A = \{z \mid 1 < |z| < 2\}$ and identify the elements of $\widetilde{[n]}$ with certain points of the boundary of A via

$$k \mapsto 2e^{\frac{2\pi i(k-1)}{n}} \quad \text{and} \quad k' \mapsto e^{\frac{2\pi i(k-1)}{n}} \quad \text{for all} \quad k \in [n].$$

For $\xi \in \mathfrak{B}_n$ take a copy of A and link any $x, y \in \widetilde{[n]}$ with $\{x, y\} \in \xi$ by a path (called *string*) running entirely in A (except for its endpoints). For example, the element $\xi \in \mathfrak{B}_4$ given by

$$\xi = \{\{1, 1'\}, \{2, 4\}, \{3, 2'\}, \{3', 4'\}\}$$

is then represented by the annular diagram in Fig. 3. Paths representing

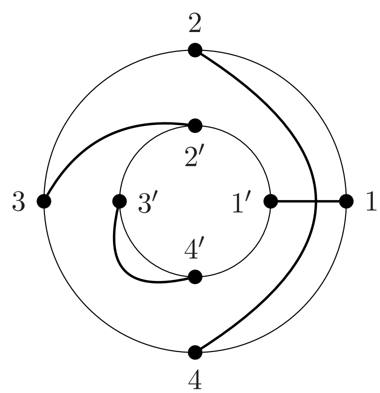


FIGURE 3. Annular diagram representation of a member of $\mathfrak{B}_4$

blocks of the form $\{x, y'\}$ [$\{x, y\}$ and $\{x', y'\}$, respectively] for some $x, y \in [n]$ are called *through strings* [*outer* and *inner strings*, respectively]. The *annular monoid* $\mathfrak{A}_n$ by definition consists of all elements of $\mathfrak{B}_n$ that have a representation as an annular diagram any two of whose strings have empty intersection. One can compose annular diagrams in an obvious way, modelling the multiplication in $\mathfrak{B}_n$ — from this it follows easily that $\mathfrak{A}_n$ is closed under the multiplication of $\mathfrak{B}_n$. Clearly, $\mathfrak{A}_n$ is closed under the involution *, as well.

Although we shall not need it, we remark that the rank characterizes the $\mathscr{D}$-relation in $\mathfrak{A}_n$, as well. Let $\xi \in \mathfrak{B}_n$ be of rank t with through strings

$$\{k_1, l_1'\}, \ldots, \{k_t, l_t'\}, \text{ for some } k_i, l_i \in [n].$$

Then $\{k_1, \ldots, k_t\}$ respectively $\{l_1', \ldots, l_t'\}$ is the *domain* $\operatorname{dom} \xi$ respectively *range* $\operatorname{ran} \xi$ of ξ. For any projection ε we obviously have $\operatorname{ran} \varepsilon = (\operatorname{dom} \varepsilon)'$.

In order to show that the rank function characterizes the $\mathscr{D}$-relation it is sufficient to show that any two projections ε, η of the same rank t are $\mathscr{D}$-related. Let ε and η be arbitrary projections of rank t with $a_1 < a_2 < \cdots < a_t$ the domain of ε and $b'_1 < b'_2 < \cdots < b'_t$ the range of η; define α to be the element having the same outer strings as ε, the same inner strings as η and the through strings

$$\{a_1, b'_1\}, \ldots, \{a_t, b'_t\}.$$

Then $\alpha \in \mathfrak{A}_n$, $\varepsilon = \alpha\alpha^*$ and $\eta = \alpha^*\alpha$.

3.2. **Subgroups in annular monoids.** It was observed by Jones [16] that the maximal subgroup of $\mathfrak{A}_n$ whose identity is a projection ε of rank t is a cyclic group of order t. Indeed, suppose that $k_1 < k_2 < \cdots < k_t$ are the elements of $\operatorname{dom}\varepsilon$ and let $\xi \in \mathfrak{A}_n$ be $\mathscr{H}$-related with ε. In ξ there exists a unique through string of the form $\{k_1, k'_\ell\}$. From the annular condition it follows that the remaining $t-1$ through strings of ξ are precisely

$$\{k_2, k'_{\ell+1}\}, \ldots, \{k_{t-\ell+1}, k'_t\},\ \{k_{t-\ell+2}, k'_1\} \ldots, \{k_t, k'_{\ell-1}\}$$

while the inner and outer strings of ξ are those of ε. Taking into account (3.3), we observe that $\xi = \tau^{\ell-1}$, where τ consists of the through strings

$$\{k_1, k'_2\}, \{k_2, k'_3\}, \ldots, \{k_t, k'_1\}$$

together with the inner and outer strings of ε. Obviously, $\tau^t = \varepsilon$ and the order of τ is t.

In the following we shall obtain some facts about $\mathrm{H}(\mathfrak{A}_n)$ in case n is even.

Lemma 3.2. *If $\{i, j\}$ [respectively $\{i', j'\}$] is an outer [inner] string of an element in $\mathfrak{A}_{2m}$, then $j - i$ is odd.*

Proof. Suppose that $j > i$. On the circuit, only points either between i and j or between j and i (in, say, counterclockwise orientation) can be involved in through strings. In the first case, all points between j and i must be involved in outer strings hence the number of points between j and i is even; since the total number of points is even, this implies that $j - i$ is odd. In the second case, by the same reason there must be an even number of points between i and j whence $j - i$ is odd, anyway. □

This enables us to obtain the next result.

Lemma 3.3. *Let $\alpha = \varepsilon_1 \cdots \varepsilon_k \in \mathfrak{A}_{2m}$ be a product of projections ε_s. Then for each through string $\{i, j'\}$ of α, the difference $j - i$ is even.*

Proof. The proof is by induction on k, the case $k = 1$ being trivial. So, suppose that $k > 1$ and set $\beta = \varepsilon_1 \ldots \varepsilon_{k-1}$ and $\varepsilon = \varepsilon_k$. Let $\{i, j'\}$ be a through string in $\beta\varepsilon$. By the definition of the multiplication, there exist $k_0, k_1, \ldots, k_{2t} \in [n]$ such that

$$i\ \beta\ k'_0,\ k_0\ \varepsilon\ k_1,\ k'_1\ \beta\ k'_2, \ldots, k'_{2t-1}\ \beta\ k'_{2t},\ k_{2t}\ \varepsilon\ j'.$$

Since ε is a projection and $\{k_{2t}, j'\}$ is a through string, we have $k_{2t} = j$. Hence

$$j - i = \sum_{s=1}^{2t}(k_s - k_{s-1}) + (k_0 - i).$$

By the induction assumption, $k_0 - i$ is even since $\{i, k_0'\}$ is a through string in β. By Lemma 3.2, each $k_{s+1} - k_s$ is odd. Consequently, the sum $\sum_{s=1}^{2t}(k_s - k_{s-1})$, containing an even number of odd summands, is even. □

For the next result note that for any $\xi \in \mathfrak{A}_{2m}$ of rank t, if $k_1 < k_2 < \cdots < k_t$ are the elements of the domain of ξ, then $k_{i+1} - k_i$ is odd for each $i < t$ — the argument is similar to that in Lemma 3.2. Since t necessarily is even, $k_t - k_1$ is also odd.

Corollary 3.4. *Let $\alpha \in \mathfrak{A}_{2m}$ be a product of projections, α having domain $k_1 < k_2 < \cdots < k_t$ and range $k_1' < k_2' < \cdots < k_t'$. If $\{k_i, k_j'\}$ is a through string of α, then $j - i$ is even.*

Proof. Suppose that $j > i$; by Lemma 3.3, $k_j - k_i$ is even. Since

$$k_j - k_i = (k_j - k_{j-1}) + \cdots + (k_{i+1} - k_i) \tag{3.4}$$

and each summand on the right hand side of (3.4) is odd by the above remark, there must be an even number of summands in that sum and that number coincides with $j - i$. □

This allows us to obtain the principal result in this context.

Corollary 3.5. *For each even number n, each maximal subgroup of* $\mathrm{H}(\mathfrak{A}_n)$ *has order less than $\frac{n}{2}$.*

Proof. Each non-trivial maximal subgroup of $\mathrm{H}(\mathfrak{A}_n)$ is isomorphic to the group $\mathscr{H}$-class (in $\mathrm{H}(\mathfrak{A}_n)$) of some projection $\varepsilon \neq 1$. Let t be the rank of ε (which is even) and $k_1 < k_2 < \cdots < k_t$ be the elements of the domain of ε. Let $\tau \in \mathfrak{A}_n$ be defined by having the through strings $\{k_1, k_2'\}, \ldots, \{k_t, k_1'\}$ and all inner and outer strings the same as ε (that is, τ is $\mathscr{H}$-related in $\mathfrak{A}_n$ to ε). Clearly, τ has order t and generates the group $\mathscr{H}$-class of ε in $\mathfrak{A}_n$. Now let $\alpha \in \mathrm{H}(\mathfrak{A}_n)$ be a generating element of the group $\mathscr{H}$-class of ε in $\mathrm{H}(\mathfrak{A}_n)$; then α and ε have the same domain and range. There exists a unique s such that $\{k_1, k_{s+1}'\}$ is a through string in α. By Corollary 3.4, s is even. By the annular condition, all the remaining through strings of α are of the form

$$\{k_2, k_{s+2}'\}, \ldots, \{k_{t-s}, k_t'\},\ \{k_{t-s+1}, k_1'\} \ldots, \{k_t, k_s'\}.$$

Then

$$\alpha = \tau^s = (\tau^2)^{\frac{s}{2}}$$

whence α lies in the subgroup generated by τ^2. The order of τ^2 is $\frac{t}{2}$, hence the order of α is at most $\frac{t}{2}$ which is less than $\frac{n}{2}$. □

Altogether we are able to detect a group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$:

Corollary 3.6. *For each even number n there exists a group in* $\mathsf{var}\,\mathfrak{A}_n$ *that is not in* $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$.

Proof. We have seen that all cyclic groups in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ have even order less than $\frac{n}{2}$. On the other hand, each cyclic group of even order up to n belongs to $\mathsf{var}\,\mathfrak{A}_n$. In order to find a (cyclic) group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ it suffices to find an even number $k \leq n$ that does not divide the least common multiple of all even numbers less than $\frac{n}{2}$. For such k we may take the largest power of 2 which is less than or equal to n. □

Since, for any n, all maximal subgroups of $\mathrm{H}(\mathfrak{A}_n)$ have order at most $n-2$, by the same reasoning as in Corollary 3.6, the next result is immediate.

Corollary 3.7. *Let n be a prime power; then the cyclic group of order n belongs to* $\mathsf{var}\,\mathfrak{A}_n$ *but not to* $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$.

Finally, we show that the cases n even or a prime power are the only ones for which there is a group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$. Therefore, our methods are applicable precisely in these cases.

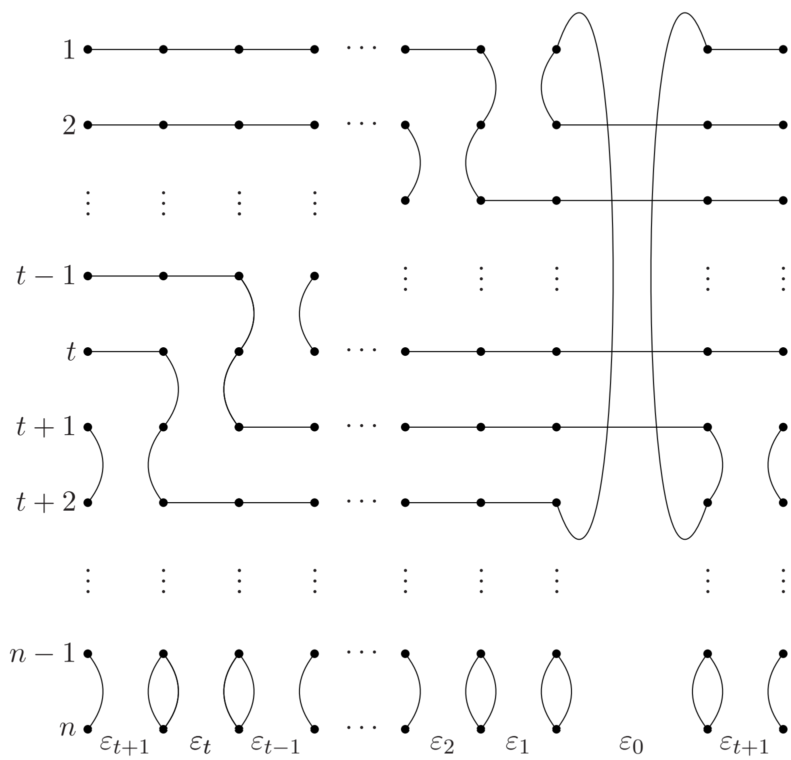


FIGURE 4. A cycle of order t as a product of projections in $\mathfrak{A}_n$

Proposition 3.8. *If n is odd and not a prime power then the groups in* $\mathsf{var}\,\mathfrak{A}_n$ *and* $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ *are the same.*

Proof. Every element of $\mathfrak{A}_n$ has odd rank. Let $t < n$ be odd. We define projections $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{t+1}$ as follows. Each ε_i, $i = 0, 1, \dots, t+1$, has the outer strings $\{t+3, t+4\}, \dots, \{n-1, n\}$ and the corresponding inner strings; besides those, the projection ε_0 has the outer string $\{1, t+2\}$ and the inner string $\{1', (t+2)'\}$ and each of the projections ε_i, $i = 1, \dots, t+1$, has the outer string $\{i, i+1\}$ and the outer string $\{i', (i+1)'\}$. Finally, all remaining elements of $\widetilde{[n]}$ are involved in the through strings $\{k, k'\}$.

One then readily verifies (see Fig. 4) that $\alpha = \varepsilon_{t+1}\varepsilon_t \cdots \varepsilon_0\varepsilon_{t+1}$ has the same inner and outer strings as ε_{t+1} hence $\alpha \mathrel{\mathscr{H}} \varepsilon_{t+1}$. Moreover, the through strings of α are

$$\{1, 3'\}, \{2, 4'\}, \dots, \{t-1, 1'\}, \{t, 2'\}.$$

Via (3.3), α realizes the cyclic permutation $k \mapsto k + 2 \pmod{t}$ which has order t since t is odd. Thus, for each odd $t < n$, the monoid $\mathrm{H}(\mathfrak{A}_n)$ contains a cyclic group of order t as a unary subsemigroup. The maximal subgroups of $\mathfrak{A}_n$ are precisely the cyclic groups of odd order at most n. So, we have already shown that $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ contains each maximal subgroup of $\mathfrak{A}_n$, with the possible exception of the group of units of $\mathfrak{A}_n$ which is cyclic of order n. Since n is not a prime power, $n = k\ell$ for some co-prime numbers k, ℓ. As already pointed out, the cyclic groups $\mathcal{C}_k$ and $\mathcal{C}_\ell$ of orders k and ℓ, respectively, belong to $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ whence so does the cyclic group of order n which is isomorphic to $\mathcal{C}_k \times \mathcal{C}_\ell$. □

Corollary 3.9. *There exists a group in* $\mathsf{var}\,\mathfrak{A}_n$ *that is not in* $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ *if and only if* n *is even or a prime power.*

Analogously to the partial Brauer monoid $P\mathfrak{B}_n$, one could also define the *partial* annular monoid $P\mathfrak{A}_n$ by considering all elements of $P\mathfrak{B}_n$ which admit a representation by an annular diagram in which any two distinct strings have empty intersection. Clearly, for each n, there is a unary semigroup embedding $P\mathfrak{A}_n \hookrightarrow P\mathfrak{A}_{n+1}$. The elements of $P\mathfrak{A}_n$ can have rank t for each $t \le n$ and the maximal subgroups of $P\mathfrak{A}_n$ are precisely the cyclic groups of orders at most n.

In contrast to the ordinary annular case, it is no longer true that there is a group in $\mathsf{var}\,P\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$ for each even n. The fact that some elements of $\widetilde{[n]}$ need not be involved in any string gives the projections more freedom to gain cyclic permutations in $\mathrm{H}(P\mathfrak{A}_n)$, as the following result demonstrates.

Proposition 3.10. *For each* $n \ge 5$ *and* $t \le n-3$ *there exists a cyclic subgroup of order* t *in* $\mathrm{H}(P\mathfrak{A}_n)$.

Proof. For odd t this follows immediately from Proposition 3.8. For even t it can be shown that the element α consisting of the through strings

$$\{1, 2'\}, \{2, 3'\}, \dots, \{t-1, t'\}, \{t, 1'\}$$

along with the outer string $\{t+2, t+3\}$ and the inner string $\{(t+2)', (t+3)'\}$, and else having no other strings can be written as a product of $\frac{5t}{2} + 4$ projections, see Fig. 5. Clearly, α realizes a cyclic permutation of order t. $\square$

From this we obtain:

Corollary 3.11. *If $n \notin \{p^k, p^k+1, 2^k+2\}$ for each prime p and each $k \geq 1$, then $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$ and $\mathsf{var}\, P\mathfrak{A}_n$ contain the same groups.*

Proof. We may assume that $n \geq 15$. As already mentioned, the variety of all groups in $\mathsf{var}\, P\mathfrak{A}_n$ is generated by all cyclic groups of orders at most n. By Proposition 3.10, all cyclic groups of orders at most $n-3$ belong to $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$. Since n is not a prime power it can be factored as $n = k\ell$ with k, ℓ co-prime and $k, \ell \leq n-3$. Since the cyclic groups of order k and ℓ belong to $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$, so does the cyclic group of order n. The same reasoning applies to the cyclic group of order $n-1$. Consider finally the case of $n-2$. By assumption, $n-2$ either is an odd prime power or has at least two distinct prime factors. In the former case the claim follows from the proof of Proposition 3.8 and in the latter case the argument is the same as for n and $n-1$. $\square$

On the other hand, the converse of Corollary 3.11 also holds.

Proposition 3.12. *If $n \in \{p^k, p^k+1, 2^k+2\}$ for some prime p and some positive integer k, then there exists a group in $\mathsf{var}\, P\mathfrak{A}_n$ which is not in $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$.*

Proof. The case $n = p^k$ is obvious. Since the product of any two *distinct* projections of rank $n-1$ has rank less than $n-1$, the group $\mathscr{H}$-class in $\mathrm{H}(P\mathfrak{A}_n)$ of any projection of rank $n-1$ is trivial, implying the claim for the case $n = p^k + 1$.

Finally, in case $n = 2^k+2$ we show that the cyclic group of order $2^k = n-2$ is not in $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$. Let ε be a projection of rank $n-2$ containing the outer string $\{i, i+1\}$ and the inner string $\{i', (i+1)'\}$ and let ε° be the projection obtained from ε by removing these two strings. If η is a projection of rank $n-1$ such that $\eta\varepsilon$ has rank $n-2$, then $\eta\varepsilon = \varepsilon^\circ\varepsilon$ (and likewise $\varepsilon\eta = \varepsilon\varepsilon^\circ$). Hence, if α is of rank $n-2$ and a product of projections then we may assume that all these projections have rank $n-2$. Moreover, any product of two distinct projections of rank $n-2$ that have only through strings has rank less than $n-2$. Finally, let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be projections of rank $n-2$ such that ε_1 and ε_3 have outer and inner strings but ε_2 does not. If $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n-2$ then $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_1\varepsilon_3 = \varepsilon_1$.

Let α be of rank $n-2$ and assume that it is a product of projections: $\alpha = \varepsilon_0\varepsilon_1 \cdots \varepsilon_r$. The observations in the preceding paragraph imply that in addition we may assume that ε_i has rank $n-2$ for each $i = 0, \ldots, r$ and that $\varepsilon_1, \ldots, \varepsilon_{r-1}$ have inner and outer strings, that is, $\varepsilon_1, \ldots, \varepsilon_{r-1}$ belong to $\mathfrak{A}_n$. Assume further that α is contained in a subgroup of $\mathrm{H}(P\mathfrak{A}_n)$. We intend to prove that the order of α is at most $\frac{n-2}{2} = 2^{k-1}$. For this purpose we may

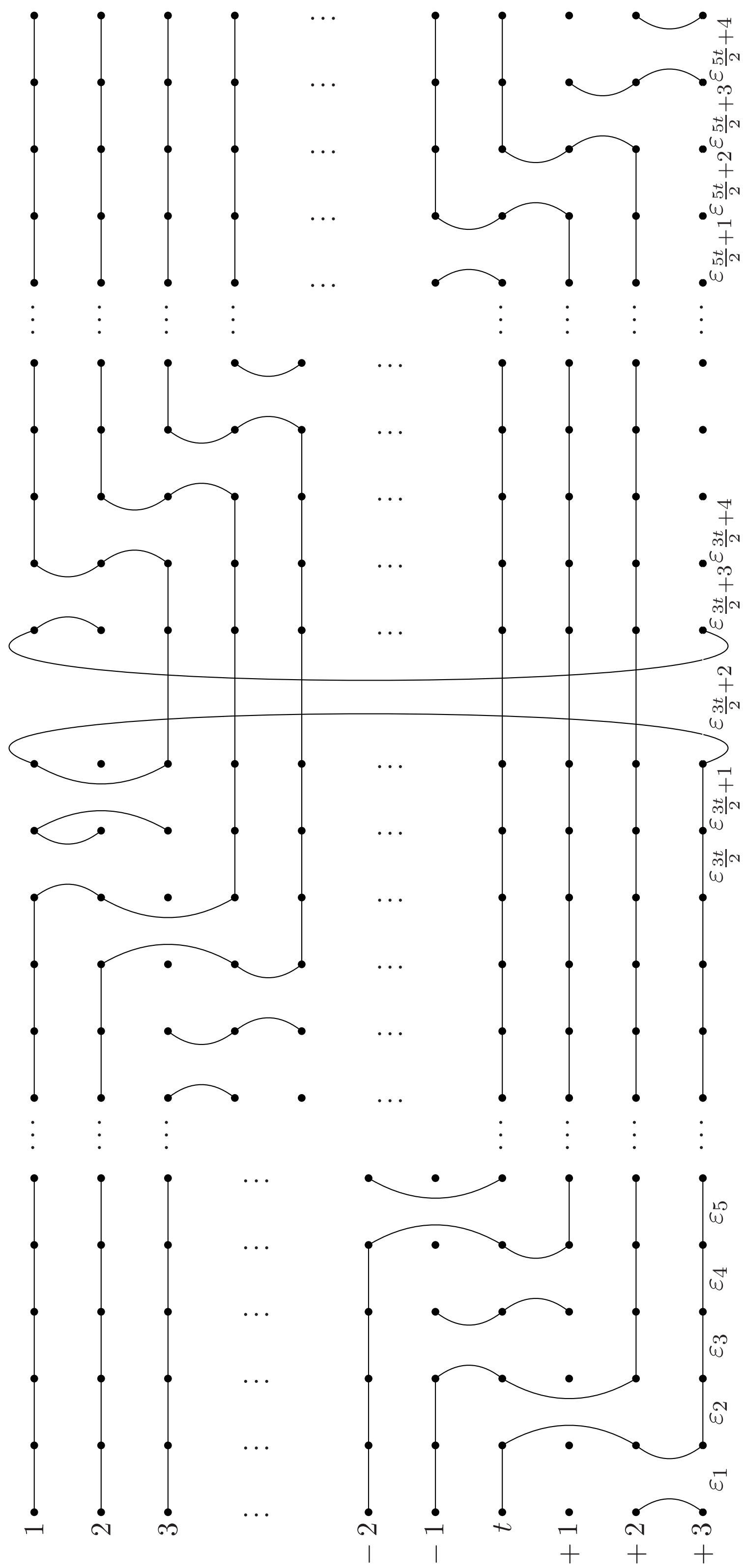


FIGURE 5. A cycle of order t as a product of projections in $P\mathfrak{A}_n$

assume that α is $\mathscr{H}$-related to a projection ε. This implies immediately that $\varepsilon_0 = \varepsilon = \varepsilon_r$.

Now consider two cases: (i) ε has an inner and an outer string, that is, ε belongs to $\mathfrak{A}_n$, and (ii) ε has only through strings, that is, ε does not belong to $\mathfrak{A}_n$. In the first case, α belongs to $\mathrm{H}(\mathfrak{A}_n)$ and so the order of α is at most $\frac{n-2}{2}$ by Corollary 3.5.

In the second case, we get $\varepsilon_1 = \varepsilon_{r-1}$ and $\varepsilon = \varepsilon_1^\circ$ since $\varepsilon\varepsilon_1$ as well as $\varepsilon_{r-1}\varepsilon$ have rank $n-2$. From this it follows that the set $\{\varepsilon, \varepsilon_1, \varepsilon\varepsilon_1, \varepsilon_1\varepsilon\}$ forms a 2×2-rectangular band under multiplication. In particular, ε and ε_1 are $\mathscr{D}$-related in $\mathrm{H}(P\mathfrak{A}_n)$. Green's Lemma implies that the order of α is the same as the order of $\varepsilon_1\alpha\varepsilon_1 = \varepsilon_1 \cdots \varepsilon_{r-1}$. The latter element belongs to $\mathrm{H}(\mathfrak{A}_n)$, so its order is at most $\frac{n-2}{2}$, again by Corollary 3.5. Since no group element of rank less than $n-2$ can have order $n-2$ we actually have shown that $\mathrm{H}(P\mathfrak{A}_n)$ does not contain a group element of order $n-2 = 2^k$.

Altogether, the cyclic group of order 2^k belongs to $\mathsf{var}\, P\mathfrak{A}_n$ but not to $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$, as required. □

3.3. **Membership of $\mathcal{K}_3$.** In order to complete the results we need to check membership of $\mathcal{K}_3$.

Proposition 3.13. *The unary semigroup $\mathcal{K}_3$ is contained in*

(1) $\mathsf{var}\, \mathfrak{C}_n$ *for each* $n \geq 2$,
(2) $\mathsf{var}\, P\mathfrak{A}_n \subseteq \mathsf{var}\, P\mathfrak{B}_n$ *for each* $n \geq 3$,
(3) $\mathsf{var}\, \mathfrak{A}_n \subseteq \mathsf{var}\, \mathfrak{B}_n$ *for each* $n \geq 4$.

Proof. In the first case, consider the unary subsemigroup $\mathcal{U}_1$ of $\mathfrak{C}_2$ generated by the projections of rank 1 — these are

$$\{\{1, 1', 2, 2'\}\}, \{\{1, 1'\}, \{2\}, \{2'\}\}, \{\{1\}, \{1'\}, \{2, 2'\}\}.$$

It is easy to calculate that $\mathcal{U}_1$ contains 13 partitions: 9 of rank 1 and 4 of rank 0. The $\mathscr{D}$-class of $\mathcal{U}_1$ consisting of partitions of rank 1 is shown in Fig. 6 where the idempotents are marked with $\star$.

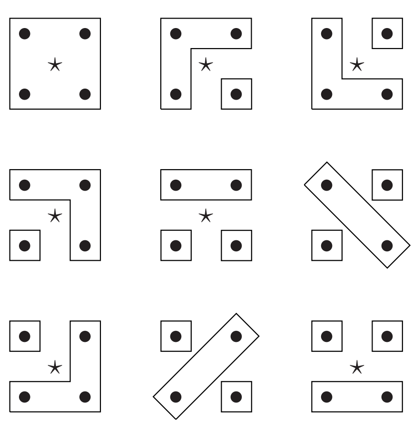


FIGURE 6. The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_1$ of $\mathfrak{C}_2$

Now it is clear that if one factors $\mathcal{U}_1$ by the ideal of all elements of rank 0, then the resulting unary semigroup is isomorphic to $\mathcal{K}_3$. Thus, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{C}_2$, and hence, to the variety $\mathsf{var}\,\mathfrak{C}_n$ for each $n \geq 2$.

For the second case consider the unary subsemigroup $\mathcal{U}_2$ of $P\mathfrak{A}_3$ generated by the projections

$$\{\{1,1'\},\{2,3\},\{2',3'\}\},$$
$$\{\{1,2\},\{1',2'\},\{3,3'\}\},$$
$$\{\{1,1'\},\{2\},\{2'\},\{3\},\{3'\}\}.$$

Again it is easy to calculate that $\mathcal{U}_2$ contains 9 partitions of rank 1 and 9 partitions of rank 0. The partitions of rank 1 form a regular $\mathscr{D}$-class depicted in Fig. 7. As above, the idempotents are marked with $\star$.

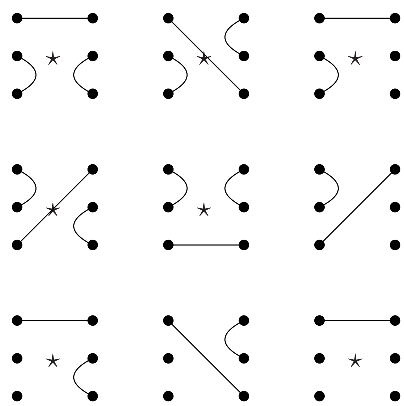


FIGURE 7. The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_2$ of $P\mathfrak{A}_3$

Again, it follows that the quotient of $\mathcal{U}_2$ by the ideal of all elements of rank 0 is isomorphic to $\mathcal{K}_3$. We see that $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,P\mathfrak{A}_3$, and hence, to the varieties $\mathsf{var}\,P\mathfrak{A}_n$ and $\mathsf{var}\,P\mathfrak{B}_n$ for each $n \geq 3$.

Finally, for the third case consider the unary subsemigroup $\mathcal{U}_3$ of $\mathfrak{A}_4$ generated by the projections

$$\{\{1,1'\},\{2,3\},\{2',3'\},\{4,4'\}\},$$
$$\{\{1,1'\},\{2,2'\},\{3,4\},\{3',4'\}\},$$
$$\{\{1,2\},\{1',2'\},\{3,3'\},\{4,4'\}\}.$$

It can be easily shown that $\mathcal{U}_3$ contains 13 partitions: 9 of rank 2 and 4 of rank 0. (Actually, one can observe that $\mathcal{U}_3$ is isomorphic to $\mathcal{U}_1$ where the isomorphism is induced by the following mapping of the base sets: $1,2 \mapsto 1$, $3,4 \mapsto 2$, $1',2' \mapsto 1'$ and $3',4' \mapsto 2'$.) Fig. 8 presents the top $\mathscr{D}$-class of $\mathcal{U}_3$ consisting of the partitions of rank 2.

Thus, factoring $\mathcal{U}_3$ by the ideal of all elements of rank 0, one gets a unary semigroup isomorphic to $\mathcal{K}_3$. Therefore, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{A}_4$,

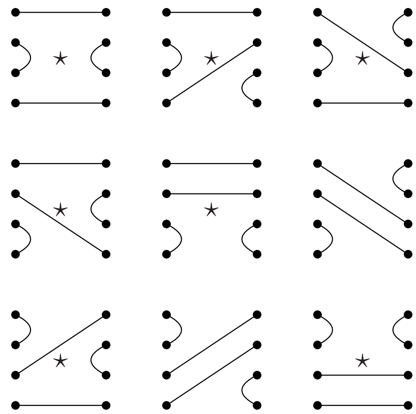


FIGURE 8. The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_3$ of $\mathfrak{A}_4$

and hence, to the varieties $\mathsf{var}\,\mathfrak{A}_n$ for all even $n \geq 4$ (recall that there is a unary semigroup embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+2}$) and $\mathsf{var}\,\mathfrak{B}_n$ for each $n \geq 4$.

It remains to verify that $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{A}_5$ (as then it also belongs to all the varieties $\mathsf{var}\,\mathfrak{A}_n$ with odd $n \geq 5$). Here an obvious modification of the above construction works, namely, we add to each of the 13 partitions forming $\mathcal{U}_3$ the new through string $\{5, 5'\}$. It is easy to see that the resulting 13 partitions lie in $\mathfrak{A}_5$ and form a unary subsemigroup isomorphic to $\mathcal{U}_3$. □

We can summarize the results of this section as follows.

Theorem 3.14. *The following regular $*$-semigroups are not finitely based:*
1. $\mathfrak{C}_n$ *for* $n \geq 2$,
2. $P\mathfrak{B}_n$ *for* $n \geq 3$,
3. $\mathfrak{B}_n$ *for* $n \geq 4$,
4. $\mathfrak{A}_n$ *for* $n \geq 4$, *n even or a prime power,*
5. $P\mathfrak{A}_n$ *for* $n \geq 3$, *n of the form* $2^k + 2$, p^k *or* $p^k + 1$ *for a prime p and* $k \geq 1$.

Proof. From Corollaries 3.1, 3.6, and 3.7 and Propositions 3.12 and 3.13, it follows that Theorem 1.2 applies in each case. □

Given this result, the question arises what happens in the cases not covered by Theorem 3.14. First of all, we may formulate

Problem 3.1. (1) Is $\mathfrak{A}_n$ finitely based for n odd, not a prime power?
(2) Is $P\mathfrak{A}_n$ finitely based for $n \notin \{2^k + 2, p^k, p^k + 1\}$ (p prime, $k \geq 1$)?

Remaining are now only some cases for small n. In case $n = 1$ we have: $\mathfrak{B}_1 \cong \mathfrak{A}_1$ is the trivial monoid which is of course finitely based and $P\mathfrak{B}_1 \cong P\mathfrak{A}_1$ is the two element semilattice (with trivial involution) which is also finitely based. In case $n = 2$, $\mathfrak{B}_2 \cong \mathfrak{A}_2$ is a Clifford semigroup (a cyclic group of order 2 with zero adjoined) which is finitely based, and $P\mathfrak{B}_2 \cong P\mathfrak{A}_2$

which turns out to be an ideal extension of a 2×2 rectangular band (with involution) by the symmetric inverse semigroup of rank 2 — we do not know if this is finitely based. Finally, in case $n = 3$ we observe that $\mathfrak{B}_3$ is an ideal extension of a 3×3 rectangular band (with involution) by the symmetric group $\mathfrak{S}_3$ and $\mathfrak{A}_3$ is an ideal extension of a 3×3 rectangular band (with involution) by the cyclic group of order 3 — in neither case we know the answer. So we may formulate

Problem 3.2. Are the regular $*$-semigroups $P\mathfrak{B}_2 \cong P\mathfrak{A}_2$, $\mathfrak{B}_3$, $\mathfrak{A}_3$ finitely based?

4. Further applications

4.1. **Unary Rees matrix semigroups.** We had used unary Rees matrix semigroups as a tool in the proof of our first main result in Section 1; in turn, here we shall show that Theorem 1.2 allows one to solve the finite basis problem for a large family of unary Rees matrix semigroups.

An $I \times I$-matrix $P = (p_{ij})$ over $\mathcal{G} \cup \{0\}$, where $\mathcal{G}$ is a group, is called *block-diagonalizable* if there exists a partition π of the set I such that $p_{ij} \neq 0$ if and only if $i\ \pi\ j$. If one defines a graph $\Gamma(P)$ on the set I in which two distinct vertices i and j are adjacent if and only if $p_{ij} \neq 0$, then it is clear that block-diagonalizable matrices correspond to graphs whose connected components are cliques (i.e. complete graphs). We say that P is $*$-*regular* if for all $i, j \in I$, one has $p_{ji} = p_{ij}^{-1}$ whenever $p_{ij} \in \mathcal{G}$ and $p_{ii} = e$, where e is the identity element of $\mathcal{G}$. (Recall that this property ensures that the unary Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is a regular $*$-semigroup.)

Theorem 4.1. *Let P be an $I \times I$-matrix over $\mathcal{G} \cup \{0\}$, where $\mathcal{G}$ is a group. Suppose that P is $*$-regular and not block-diagonalizable. If $\mathcal{G}$ does not belong to the group variety $\mathsf{var}\,\mathcal{H}$, where $\mathcal{H}$ is the subgroup generated by the non-zero entries of P, then the unary Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is not finitely based.*

Proof. Let $P = (p_{ij})$. Since P is not block-diagonalizable, there is a connected component C in $\Gamma(P)$ which is not a clique. Let Q be a maximal clique in C. As C is connected, there exist $i_0 \in Q$ and $j_0 \in C \setminus Q$ such that i_0 and j_0 are adjacent. At the same time, there should be a vertex $k_0 \in Q$ such that j_0 and k_0 are not adjacent — otherwise $Q \cup \{j_0\}$ would make a larger clique in C. Thus, the submatrix P_0 of P corresponding to the set $I_0 = \{i_0, j_0, k_0\}$ is of the form

$$P_0 = \begin{pmatrix} e & g & h \\ g^{-1} & e & 0 \\ h^{-1} & 0 & e \end{pmatrix}$$

where $g = p_{i_0 j_0}, h = p_{i_0 k_0}$ belong to $\mathcal{G}$. The unary Rees matrix semigroup $\mathcal{M}^0(I_0, \mathcal{G}, I_0; P_0)$ is then a unary subsemigroup in $\mathcal{M}^0(I, \mathcal{G}, I; P)$, and the obvious homomorphism $\mathcal{G} \to \mathcal{E}$, where $\mathcal{E} = \{e\}$ is the trivial group, extends

to a unary semigroup homomorphism from $\mathcal{M}^0(I_0, \mathcal{G}, I_0; P_0)$ onto $\mathcal{K}_3$. Thus, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{G}, I; P)$.

The Hermitian subsemigroup $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ of $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is generated by the elements (i, e, i) where i runs over I. This implies that the group coordinates of triples (i, g, j) in $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ belong to the subgroup $\mathcal{H}$ generated by the non-zero entries of P. Hence $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ is a unary subsemigroup of the unary Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{H}, I; P)$. It is not hard to see that each group in the variety $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{H}, I; P)$ belongs to the group variety $\mathsf{var}\,\mathcal{H}$. Since $\mathcal{G}$ does not belong to $\mathsf{var}\,\mathcal{H}$ but obviously belongs to $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{G}, I; P)$, we are in a position to apply Theorem 1.2. □

A comprehensive treatment of the finite basis problem for unary Rees matrix semigroups forms the subject of a forthcoming paper by Jackson and the third author.

4.2. **Varietal joins.** Recall that the *join* $\mathbf{V} \vee \mathbf{W}$ of two varieties $\mathbf{V}$ and $\mathbf{W}$ is the least variety containing both $\mathbf{V}$ and $\mathbf{W}$. We show how Theorem 1.2 can be used to produce interesting examples of non-finitely based joins of varieties of unary semigroups.

Denote by $\mathbf{CSR}^*$ the variety generated by the unary semigroup $\mathcal{K}_3$ (that is, the variety of all *combinatorial strict regular $*$-semigroups*, see [3]) and let $\mathbf{I}$ be the variety of all inverse semigroups.

Theorem 4.2. *Let $\mathbf{K}$ and $\mathbf{A}$ be varieties of unary semigroups such that:*

(1) *$\mathbf{K}$ contains $\mathbf{CSR}^*$,*
(2) *$\mathbf{A}$ consists of inverse semigroups and contains a group not contained in $\mathrm{H}(\mathbf{K})$.*

Then no variety in the interval $[\mathbf{CSR}^* \vee \mathbf{A}, \mathbf{K} \vee \mathbf{I}]$ is finitely based.

Proof. Let $\mathcal{S} \in \mathbf{K} \vee \mathbf{I}$; then there exist $\mathcal{K} \in \mathbf{K}$, $\mathcal{I} \in \mathbf{I}$ such that $\mathcal{S}$ *divides* (that is, $\mathcal{S}$ is a homomorphic image of a substructure of) $\mathcal{K} \times \mathcal{I}$ whence $\mathrm{H}(\mathcal{S})$ divides $\mathrm{H}(\mathcal{K}) \times \mathrm{H}(\mathcal{I})$. Observe that $\mathrm{H}(\mathcal{I})$ is a semilattice (with trivial involution). Further, since $\mathrm{H}(\mathcal{K}_3) = \mathcal{K}_3$ and $\mathbf{CSR}^* \subseteq \mathbf{K}$ we have $\mathbf{CSR}^* = \mathrm{H}(\mathbf{CSR}^*) \subseteq \mathrm{H}(\mathbf{K})$ so that $\mathrm{H}(\mathbf{K})$ contains all semilattices with trivial involution since $\mathbf{CSR}^*$ does so. Altogether, we have $\mathrm{H}(\mathcal{S}) \in \mathrm{H}(\mathbf{K})$, that is $\mathrm{H}(\mathbf{K} \vee \mathbf{I}) = \mathrm{H}(\mathbf{K})$, and thus, for any variety $\mathbf{V}$ in the interval $[\mathbf{CSR}^* \vee \mathbf{A}, \mathbf{K} \vee \mathbf{I}]$, we have $\mathrm{H}(\mathbf{V}) \subseteq \mathrm{H}(\mathbf{K})$. By assumption (2), there exists a group in $\mathbf{A} \subseteq \mathbf{V}$ that is not in $\mathrm{H}(\mathbf{K}) \supseteq \mathrm{H}(\mathbf{V})$. Thus, Theorem 1.2 applies to the variety $\mathbf{V}$. □

The conditions of Theorem 4.2 are obviously fulfilled if $\mathbf{CSR}^* \subseteq \mathbf{K}$ and $\mathbf{A}$ contains a group that is not in $\mathbf{K}$; so, for example $\mathbf{K} = [x^m = x^{m+n}]$ for fixed $n \geq 1$ and $m \geq 2$ and $\mathbf{A} = \mathbf{G}$ (the variety of all groups) meet the requirements.

Recall that a variety $\mathbf{V}$ of algebraic structures is a *Cross variety* if

1) $\mathbf{V}$ is generated by a finite structure,
2) $\mathbf{V}$ contains only finitely many subvarieties,
3) $\mathbf{V}$ is finitely based.

For an interesting treatment of Cross varieties of plain semigroups consult Sapir [39]. The variety $\mathbf{CSR}^*$ is a Cross variety, see [3, Theorems 5.1 and 5.2, Corollary 5.4]. Now let $\mathbf{A}_p$ denote the variety of all abelian groups of exponent p (p is a prime number); clearly, $\mathbf{A}_p$ is a a Cross variety. By [3, Corollaries 5.4 and 6.5], the join $\mathbf{CSR}^* \vee \mathbf{A}_p$ contains only fourteen subvarieties; however, by the above remark, the join is not finitely based and therefore is not a Cross variety. We thus have a simple example of two Cross varieties whose join is not a Cross variety. A plain semigroup example of this kind found in [39, Corollary 2.1] is much more involved (with 39 subvarieties).

5. Existence varieties of locally inverse semigroups

In this section we give an application to existence varieties of the method of proof of Theorem 1.2. Recall that an *existence variety* (shortly *e-variety*) of regular semigroups is a class of regular semigroups closed under taking direct products, *regular* subsemigroups and homomorphic images. This section assumes the reader's acquaintance with some basics of the theory of regular semigroups.

While research into the structure of regular semigroups was particularly active in the 1970s and early 1980s, a universal algebra approach for regular semigroups has been introduced at the end of the 1980s by Kaďourek and Szendrei [17] for orthodox semigroups, and, independently, by Hall [14, 15] for regular semigroups in general. We shall recall the basic definitions and results necessary to understand the following treatment. For further information consult the papers [4, 5, 14, 15, 17, 46].

A regular semigroup $\mathcal{S} = \langle S, \cdot \rangle$ is *locally inverse* if for each idempotent e of $\mathcal{S}$, the *local submonoid* eSe is an inverse semigroup. The class $L\mathbf{I}$ of all (regular) locally inverse semigroups is a typical example of an existence variety. Observe that Rees matrix semigroups over groups are locally inverse (moreover, in such a semigroup each local submonoid is a group with 0 adjoined or the trivial group).

It is known [30, Theorem 7.6] that a regular semigroup $\mathcal{S}$ is locally inverse if and only if for any two $x, y \in \mathcal{S}$ the set $xV(yx)y$ is a singleton (as usual, $V(z)$ denotes the set of all inverses of the element z). This gives rise to the *sandwich operation* $\wedge$ that can be defined on any locally inverse semigroup by setting $x \wedge y$ to be the unique element of $xV(yx)y$, so that in this context, locally inverse semigroups are treated as algebras of type $(2, 2)$.

As explained in [4, 5], the adequate concept of equational theory for e-varieties of locally inverse semigroups is based on the signature $\{\cdot, \wedge\}$ and is with respect to a doubled alphabet $X \cup X'$. Here X is, as usual, a countably infinite set of variables and $X' = \{x' \mid x \in X\}$ is a disjoint copy of X; the elements of X' are devoted to represent inverses of the elements which are represented by the elements of X. The terms are over this extended alphabet and are in the signature $\{\cdot, \wedge\}$ where $\cdot$ stands for the associative

operation of multiplication and $\wedge$ for the sandwich operation. Given a term $w(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ of this kind, a value of that term in the locally inverse semigroup $\mathcal{S}$ is obtained by substituting the variables x_i, x'_i by elements s_i, s'_i in $\mathcal{S}$ in such a way that each s'_i is an inverse of s_i. (In this evaluation it definitely may happen that distinct variables x, y, say, will be substituted with the same element s, while the formal inverses x' and y', respectively, are substituted with distinct inverses $s^\sharp$ and $s^\flat$, say, of s.) Given this notion of evaluation of terms in a locally inverse semigroup, it is clear what it means that a locally inverse semigroup $\mathcal{S}$ *satisfies* a *bi-identity* $u = v$ of terms of that kind. The following Birkhoff type theorem then holds [4].

Theorem 5.1. *A class* $\mathbf{V}$ *of locally inverse semigroups is an e-variety if and only if it is definable by bi-identities, that is,* $\mathbf{V}$ *consists of all locally inverse semigroups that satisfy a certain set of bi-identities.*

Now call a set B of bi-identities a *basis* of $\mathbf{V}$ if a locally inverse semigroup $\mathcal{S}$ is a member of $\mathbf{V}$ if and only if $\mathcal{S}$ satisfies all bi-identities of B. This semantic notion of basis is equivalent to a syntactic one: B is a basis of $\mathbf{V}$ if and only if B axiomatizes the *bi-equational theory* which is the set of all bi-identities over a (fixed) countable infinite set X of variables satisfied by all members of $\mathbf{V}$. The latter means that each bi-identity satisfied by all members of $\mathbf{V}$ can be derived, using natural deduction rules, from the bi-identities of $B \cup B(L\mathbf{I})$ where $B(L\mathbf{I})$ is a basis for the bi-equational theory of the class of all locally inverse semigroups. A set consisting of four independent bi-identities which may serve as $B(L\mathbf{I})$ has been found in [5]. For more analogues between the theory of e-varieties of regular semigroups and varieties of universal algebras see [4, 5, 17, 46].

The objective of this section is to obtain an analogue of Theorem 1.2 giving a sufficient condition for an e-variety $\mathbf{V}$ of locally inverse semigroups to have no finite basis of bi-identities. Let $\mathcal{S}$ be a locally inverse semigroup and $a_1, \ldots, a_k \in \mathcal{S}$. For each i take an element $a'_i \in V(a_i)$. Then the closure of the set $\{a_1, \ldots, a_k, a'_1, \ldots, a'_k\}$ under multiplication and sandwich operation is a locally inverse subsemigroup of $\mathcal{S}$, and is the least locally inverse subsemigroup of $\mathcal{S}$ containing the set $\{a_1, \ldots, a_k, a'_1, \ldots, a'_k\}$ (by [46]). We call such a subsemigroup a *k-generated* locally inverse subsemigroup of $\mathcal{S}$. Define the *content* $c(t)$ of a term t inductively by $c(x) = c(x') = \{x\}$ and $c(uv) = c(u \wedge v) = c(u) \cup c(v)$. In order to prove that an e-variety $\mathbf{V}$ has no a finite basis of bi-identities it is sufficient to prove for each natural number k the existence of a locally inverse semigroup $\mathcal{T}_k$ such that $\mathcal{T}_k \notin \mathbf{V}$ but $\mathcal{T}_k$ satisfies each bi-identity $u = v$ that holds in $\mathbf{V}$ and for which $|c(u) \cup c(v)| \leq k$. The latter is equivalent to the property that each k-generated locally inverse subsemigroup (as defined above) is contained in $\mathbf{V}$.

For each e-variety $\mathbf{V}$ denote by $\mathrm{C}(\mathbf{V})$ the sub-e-variety of $\mathbf{V}$ generated by all idempotent generated members of $\mathbf{V}$. We also need the 5-element Rees matrix semigroup $\mathcal{A}_2$ with the sandwich matrix (1.8). We are ready to formulate an e-variety analogue of Theorem 1.2.

Theorem 5.2. *Let* $\mathbf{V}$ *be a locally inverse e-variety containing the semigroup* $\mathcal{A}_2$. *If* $\mathbf{V}$ *contains a group which is not in* $\mathrm{C}(\mathbf{V})$ *then* $\mathbf{V}$ *has no finite basis for its bi-identities.*

Proof. This can be proved in a manner similar to the proof of Theorem 1.2. As mentioned above, we have to prove, for each k, the existence of a locally inverse semigroup $\mathcal{T}_k$ such that $\mathcal{T}_k \notin \mathbf{V}$ but each k-generated locally inverse subsemigroup of $\mathcal{T}_k$ is contained in $\mathbf{V}$.

Let $\mathcal{G}$ be a group in $\mathbf{V}$ that is not contained in $\mathrm{C}(\mathbf{V})$. Since there must be a bi-identity which holds in $\mathrm{C}(\mathbf{V})$ but fails in $\mathcal{G}$, we may assume that $\mathcal{G}$ is generated by finitely many elements, say $g_1, \dots, g_m$, and for convenience we may assume that this set of generators is closed under taking inverse elements and so generates $\mathcal{G}$ as a semigroup. Next, let $\mathcal{T}_k$ be the Rees matrix semigroup in the proof of Theorem 1.2, but with $n = 2k + 1$ being replaced with $n = 4k+1$. In that proof it has been shown that $(1, g_j, mn) \in \langle E(\mathcal{T}_k)\rangle$ for $j = 1, \dots, m$ (here $\langle E(\mathcal{T}_k)\rangle$ denotes the idempotent generated subsemigroup of $\mathcal{T}_k$). It follows that $\{1\} \times \mathcal{G} \times \{mn\} \subseteq \langle E(\mathcal{T}_k)\rangle$ whence $\langle E(\mathcal{T}_k)\rangle$ contains a subgroup isomorphic to $\mathcal{G}$. Consequently, $\langle E(\mathcal{T}_k)\rangle \notin \mathrm{C}(\mathbf{V})$ which implies that $\mathcal{T}_k \notin \mathbf{V}$.

Finally, consider any k-generated locally inverse subsemigroup of $\mathcal{T}_k$, that is, choose $a_1, \dots, a_k, a'_1, \dots, a'_k \in \mathcal{T}_k$ such that $a'_i \in V(a_i)$ for each i. Let $\mathcal{T}$ be the locally inverse subsemigroup generated by $\{a_1, \dots, a_k, a'_1, \dots, a'_k\}$ (that is, the closure of that set under multiplication and sandwich operation). It is clear that at most $4k$ indices of $\{1, \dots, nm\}$ can occur in the triple representation of the elements a_i, a'_i. Therefore, analogously to the unary case of Section 1, there exist numbers $\lambda_1, \dots, \lambda_m$ such that

$$1 \leq \lambda_1 \leq n < \lambda_2 \leq 2n < \dots (m-1)n < \lambda_m \leq mn$$

and $\mathcal{T}$ is contained in the semigroup $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ (which is defined in the same way as in the proof of Theorem 1.2). Again as in Section 1, we can show that $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ is isomorphic to a homomorphic image of the direct product $\mathcal{G} \times \mathcal{U}_k$ of the group $\mathcal{G}$ and a completely 0-simple semigroup $\mathcal{U}_k$ with trivial subgroups. Now $\mathcal{G} \in \mathbf{V}$ by our assumption and $\mathcal{U}_k \in \mathbf{V}$ by a result of Hall [15] because $\mathbf{V}$ contains $\mathcal{A}_2$. This completes the proof. □

Theorem 5.2 can be in particular applied to certain joins of e-varieties. Here is an example. Denote by **CSR** the e-variety generated by the semigroup $\mathcal{A}_2$ and by **GI** the e-variety of all orthodox locally inverse semigroups (these semigroups are often called *generalized inverse*). The proof of the next corollary is analogous to that of Theorem 4.2 and is left to the reader.

Corollary 5.3. *Let* $\mathbf{K}$ *and* $\mathbf{A}$ *be locally inverse e-varieties such that:*

(1) $\mathbf{K}$ *contains* **CSR**,
(2) $\mathbf{A}$ *consists of orthodox semigroups and contains a group not contained in* $\mathrm{C}(\mathbf{K})$.

Then no e-variety in the interval $[\mathbf{CSR} \vee \mathbf{A}, \mathbf{K} \vee \mathbf{GI}]$ *is finitely based.*

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Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: `karl.auinger@univie.ac.at`

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: `dockie@im.ns.ac.yu`

Faculty of Mathematics and Mechanics, Ural State University, Lenina 51, 620083 Ekaterinburg, Russia
E-mail address: `mikhail.volkov@usu.ru`